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Mean-field theory of spin-liquid states with finite energy gap and topological orders*

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ABSTRACT: The mean field theory of a T and P symmetric spin liquid state is developed. The quasiparticle excitations in the spin liquid state are shown to be spin $\frac{1}{2}$ neutral fermions (the spinons) and charge e spinless bosons (the holons). The spin liquid state is shown to be characterized by a non-trivial topological order. Although our discussions are based on the mean field theory, the concept of the topological order and the associated universal properties (*e.g.,* the quantum number of the quasiparticles) are expected to be valid beyond the mean field theory. We also discuss the dynamical stability of the mean field theory.

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I. INTRODUCTION

The existence of the Mott insulator in two and higher dimensions remains to be one of the unresolved problems in theoretical physics. Here by the Mott insulator we mean an insulator with *odd* number electrons per unit cell. In past a few year this problem attracts a lot of attentions due to its relation to the high T_c superconductors.[1]

In one dimension the Mott insulator can be shown to exist thank to the exact result of the 1D Hubbard model.[2] However in higher dimensions no exact results are available. It is not clear whether the Mott insulator can exist or not. Recent mean field results strongly suggest a Mott insulator – chiral spin state[3] – may exist in two dimensions. This Mott insulator breaks time reversal symmetry (T) and parity (P).

In this paper we are going to argue that a T and P symmetric Mott insulator may exist in two and higher dimensions base on the mean field approach to spin liquid states. The Mott insulator corresponds to the short ranged resenanting-valence-bound (s-RVB) state conjectured before.[4] We will show that the Mott insulators in higher dimensions are closely related to the two known incompressible liquid states of electron systems – the quantum Hall states and the superconducting states. The quantum Hall states are directly related to the chiral spin states and, as we will see, the superconducting states is closely related to s-RVB state.

We will also discuss in detail the effect of the gauge fluctuations in the mean field theory. We show that the gauge fluctuations are very important and in many cases cause the infrared divergence in mean field states. Those mean field states are not self consistent. To construct self consistent mean field states, one needs to find a way to control the infrared divergence caused by the gauge field fluctuations. As we will see later the mean field theory of the T and P symmetric Mott insulator (or s-RVB state) has very good infrared stability due to the Higgs mechanism and is self consistent. This strongly suggest that the T and P symmetric Mott insulator is a generic state and is supported by some Hamiltonians.

Of cause, the mean field theory of spin liquid states is not reliable due to quantum fluctuations. The mean field theory can only provide some qualitative results. However the mean field theory deserves further developments because of the following reasons: A) The mean field theory does provide some insights about *possible* spin liquid states. Although it cannot determine specifically which spin Hamiltonians that actually support the spin liquids constructed, the mean field theory does provide some clue about what type of the interactions may favor the spin liquids under consideration. It also tell us the characteristic properties of those spin liquid states so that if they are discovered in numerical calculations we are able to recognize them. B) One of main problems we are going to address in this paper is the self constancy of the mean field theory. In order for the mean field theory to have any chance to describe the real spin liquids, it must have certain stability in the infrared limits. By which we mean that a good mean field ground state should be a generic state *i.e.,* be stable against any small perturbations. If such a stability exists, the mean field ground state may have some chance to survive the quantum fluctuations and to describe a real spin liquid (qualitatively). Many mean field ground states studied before do not have this infrared stability. In this paper we are going to out line sufficient conditions under which the mean field ground states are dynamically stable. In summary the mean field theory as a primitive and the only analytic approach to spin liquids in higher dimensions needs further developments to bring it closer to reality.

In general the Mott insulator described by the s-RVB states and the chiral spin states represent new kinds of universality classes of insulators. Those new universality classes of insulators are characterized by topological orders. The holes in the new insulators have unusual properties (*e.g.,* the unusal statictics). Thus the doped insulators become some sort of strange metals (*e.g.,* boson metals or semion metals). It would be very interesting to see whether those strange metals can explain the unusual normal state properties observed in the high T_c samples.

The paper is arranged as follows. In section 2 we will briefly review the mean field approach to spin liquid states and discuss infrared dynamical stability of the mean field theory. In section 3, a mean field theory of the s-RVB state is discussed. The spin excitations are found to be spin $\frac{1}{2}$ fermions (the spinons). In section 4 we study dynamical properties of the doped holes in the s-RVB state. We show that the holes are charge e bosons (the holons). The flux is shown to be quantized in unit $hc/2e$ even in the charge e holon condensed state. In section 5 we discuss how to characterize the s-RVB state. It is shown that the s-RVB state contains non-trivial topological orders.

II. MEAN FIELD THEORY OF THE SPIN LIQUID STATES AND DYNAMICAL STABILITY OF THE MEAN FIELD THEORY

In this section I will briefly review the mean field approach to the spin liquid state developed in Ref. 5,6,7.

At half filling the Hubbard model reduce to the Heseinberg model:

$$H = \sum_{\langle ij \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j \tag{1}$$

with possible frustrations. To obtain the mean field ground state of the spin liquids we introduce the slave fermion operators $s_{i\alpha}$, $\alpha = 1, 2$. The spin operator $\mathbf{S}_i$ can be expressed as

$$\mathbf{S}_i = s^\dagger_{i\alpha} \sigma_{\alpha\beta} s_{i\beta} \tag{2}$$

In terms of the slave fermion operators the Hamiltonian (1) can be rewritten as

$$H = \sum_{\langle ij \rangle} -2J_{ij} s^\dagger_{i\alpha} s_{j\alpha} s^\dagger_{j\beta} s_{i\beta} \tag{3}$$

Notice that the Hilbert space of (3) is larger than that of (1). The equivalence between (1) and (3) is valid only in the subspace where there is exactly one slave fermion per site. Therefore, in order to use (3) to describe the spin state we need to impose the following constraint

$$s^\dagger_{i\alpha} s_{i\alpha} = 1, \quad s_{i\alpha} s_{i\beta} \epsilon_{\alpha\beta} = 0 \tag{4}$$

The second constraint is actually a consequence of the first one.

Introducing the Hubbard-Stratonovich (H-S) field

$$\begin{aligned} \eta_{ij} &= \epsilon^{\alpha\beta} \; s_{i\alpha} \; s_{j\beta} = \eta_{ji} \\ \chi_{ij} &= s^\dagger_{i\alpha} s_{j\alpha} = \chi^\dagger_{ji} \end{aligned} \tag{5}$$

we may rewrite the Hamiltonian (3) as

$$
\begin{aligned}
H_{mean} = & \sum \; J_{ij} \Big[|\eta_{ij}|^2 \; + |\chi_{ij}|^2 \; - \; (\chi_{ji} \; s_{i\alpha}^\dagger s_{j\alpha} + \; \eta_{ij} \; s_{i\alpha} s_{j\beta} \; \epsilon^{\alpha\beta} + h.c.) \Big] \\
& + \sum \Big[a_0^3 (s_{i\alpha}^\dagger s_{i\alpha} - 1) + [(a_0^1 + i a_0^2) s_{i\alpha} s_{i\beta} \epsilon_{\alpha\beta} + h.c.] \Big]
\end{aligned} \tag{6}
$$

The original Hamiltonian (3) can be recovered by integrating out the H-S field η_{ij} and χ_{ij}. a_0^l are the Lagrangian multipliers to enforce the constraint (4).

The Hamiltonian (6) and the constraints (4) have a local $SU(2)$ symmetry.[7] The local $SU(2)$ symmetry becomes explicit if we introduce

$$
(\psi_{a\alpha}) = \begin{pmatrix} s_1 & s_2 \\ s_2^\dagger & -s_1^\dagger \end{pmatrix} \tag{7}
$$

$$
8 \; U_{ij} = \begin{pmatrix} -\chi_{ij}^\dagger & \eta_{ij} \\ \eta_{ij}^\dagger & \chi_{ij} \end{pmatrix} \tag{8}
$$

Using (7) and (8) we can rewrite (6) as:

$$
H_{mean} \; = \; \sum J_{ij} \; \mathrm{Tr} \Big[8 \; U_{ij}^\dagger \; U_{ij} \; + (\psi_i^\dagger U_{ij} \psi_j + \; h.c.) \Big] + \sum a_0^l \mathrm{Tr}(\psi_i^\dagger \tau^l \psi_i) \tag{9}
$$

where $\tau^l, \; l = 1, 2, 3$ are the Pauli matrices. From (9) we can see clearly that the Hamiltonian is invariant under a local $SU(2)$ gauge transformation W_i:

$$
\begin{aligned}
\psi_i \to & W_i \; \psi_i \\
U_{ij} \to & W_i \; U_{ij} \; W_j^\dagger
\end{aligned} \tag{10}
$$

A mean field ground state at "zeroth" order is obtained by making the following approximations. First we replace constraint (4) by its vacuum average

$$
\langle s_{i\alpha}^\dagger s_{i\alpha} \rangle = 1, \quad \langle s_{i\alpha} s_{i\beta} \epsilon_{\alpha\beta} \rangle = 0 \tag{11}
$$

Such a constraint can be enforced by the *site dependent* Lagrangian multipliers $a_0^l(i)$ in the Hamiltonian. At the zeroth order we ignore the fluctuations of a_0^l, *i.e.,* assuming that a_0^l is time independent. If we included the fluctuations of the a_0^l the constraint (11) would become the original constraint (4). Second we assign specific values to η_{ij} and χ_{ij} and ignoring their quantum fluctuations. In this approximation the dynamics of the slave fermions is governed by the mean field Hamiltonian

$$
H_{mean} = \sum \; J_{ij} \; \mathrm{Tr}(\psi_i^\dagger \; U_{ij}^{(mean)} \; \psi_j + h.c.) + \sum a_0^l \mathrm{Tr}(\psi_i^\dagger \tau^l \psi_i) \tag{12}
$$

where $U_{ij}^{(mean)}$ is the mean field solution, which satisfies the self consistency condition

$$
\chi_{ij}^{(mean)} = \langle s_{i\alpha}^\dagger s_{j\alpha} \rangle, \quad \eta_{ij}^{(mean)} = \langle s_{i\alpha} s_{i\beta} \epsilon_{\alpha\beta} \rangle \tag{13}
$$

a_0^l in (12) are chosen such that (11) is satisfied.

If the fluctuations of the mean field order parameters χ_{ij} and η_{ij} had finite energy gaps, it would be qualitatively correct to ignore those fluctuations, at least at low energies. However due to the $SU(2)$ gauge symmetry the "phase" fluctuations of U_{ij}:

$$U_{ij} = U_{ij}^{(mean)}\ e^{iA_{ij}} \tag{14}$$

are gapless, where $A_{ij} = a_{ij}^l \tau^l$ is 2×2 traceless hermitian matrix. We must include those phase fluctuations in order to obtain even qualitatively correct results for spin liquid states at low energies. This lead us to the "first" order mean field theory. At the first order, we include the "phase" fluctuations of U_{ij}, *i.e.,* A_{ij}, and the fluctuations of the Lagrangian multipliers a_0^l. Those fluctuations describe a $SU(2)$ lattice gauge field.[6,7] It can be shown that the $SU(2)$ gauge field fluctuations enforce the operator constraint (4). At the first order, the slave fermion dynamics is governed by:

$$H_{mean} = \sum\ J_{ij}\ \mathrm{Tr}(\psi_i^\dagger\ U_{ij}^{(mean)} e^{ia_{ij}^l \tau^l}\ \ \psi_j + h.c.) + \sum a_0^l \mathrm{Tr}(\psi_i^\dagger \tau^l \psi_i) \tag{15}$$

(15) describes an fermion system interacts with a $SU(2)$ lattice gauge theory.

Given the above mean field formalism, we would like to have a general discussion about the mean field theory of spin liquid state. At the "zeroth" order the density of the slave fermions can still fluctuates. Some times the slave fermion density fluctuations may even be gapless. In those cases the density fluctuation is so strong that the constraint (4) is badly violated. The zeroth order mean field theory (12) may fail to provide even a qualitative description of the low energy properties of a true spin liquid state (which contains no charge fluctuations). *e.g.,* at the "zeroth" order the quasiparticle excitations are spin $\frac{1}{2}$ fermions. However after including the gauge fluctuations and imposing the constraint (4) those quasiparticles may not be able to survive and appear in the physical spectrum of the true spin liquid states.

After taking into account the $SU(2)$ gauge interaction, the density fluctuations of the slave fermions should at least have finite energy gap and the slave fermion gas should be incompressible. In this case the constraint (4) is satisfied by (15) at least in the infrared limit.

In general the gauge field will mediate a confining interaction between the quasiparticles in the "zeroth" order mean field theory. As a result the quasiparticles obtain an infinite self-energy and are confined. The confinement prevent those quasiparticles from appearing in the physical spectrum of the true spin liquid states.[8] This indicates that the quantum fluctuations are extremely important which may even qualitatively change the physical picture arisen from the zeroth order mean field theory. When the gauge interaction are confining, the "zeroth" order of the mean field theory even fail to give a qualitative description of the spin liquid states. In order to obtain a qualitative description of spin liquid states from the mean field theory, we must include the gauge fields.

However, a system of fermions with $SU(2)$ gauge interaction usually is so complicated that it is very difficult to obtain the low energy properties of the system. The difficulty comes from the gaplessness of the fermion excitations and the gauge excitations. The system has severe infrared divergence. The infrared properties of the system can hardly be determined. The mean field theory essentially "maps" a complicated spin system into a (probably more) complicated fermion system with $SU(2)$ gauge interaction. The low energy properties of both systems are hard to obtain. In this case we have to say the mean field theory is not so useful in the sense that the mean field theory fail provide qualitative informations about the low energy properties of spin liquid states.

But, there are exceptions to the above misfortune. There are three possible cases in which the low energy properties of (15) can be relatively reliably determined.

A) The mean field solution U_{ij} breaks the translation symmetry and the slave fermions described by (12) (at the "zeroth" order) form a band insulator (Here we have assumed that $\eta_{ij}=0$). In this case even at the "zeroth" order without the $SU(2)$ gauge field fluctuations, the slave fermion gas is already incompressible. Due to the finite energy gap, we can safely integrate out the electrons and thus reducing (15) into a pure lattice gauge theory. The low energy properties of original spin system can be obtained from the low energy properties of the effective lattice gauge theory, which in many cases is still non-trivial. In general the gauge field mediate a confining interaction and the spin $\frac{1}{2}$ slave fermion cannot appear in physical spectrum. Only slave fermion pairs can appear as physical quasiparticles which carry integer spins. The spin excitations in general have finite gap. The spin-Peierls state discuss in Ref. 9 is an example of this type of spin states. Here we would like to point out that the translation non-invariance of the unphysical quantities U_{ij} does not necessarily mean that the physical spin state (obtained after integrating out the $SU(2)$ gauge field or after doing the Gutzwiller projection) violates the translation symmetry. The necessary and sufficient conditions for the physical spin state to be translational invariant is that U_{ij} and the translated $U'_{ij}=U_{i+\tau,\ j+\tau}$ are gauge equivalent, *i.e.,* there exists a gauge transformation W_i such that

$$U_{ij}=W_i\ U'_{ij}\ W_j^{-1} \tag{16}$$

Therefore even though the "zeroth" order mean field theory violates the translation symmetry, the corresponding spin state might still be a spin liquid state which respects the translation symmetry. It would be interesting to find such examples.

B) The mean field solution $U_{ij}^{(mean)}$ generates a flux (again assuming $\eta_{ij}\ =0$). The slave fermions described by (12) behave as if they are moving in a magnetic field. When the filling fraction is right, an integral number of Landau levels are completely filled. The slave fermion gas is incompressible due to the finite gap between Landau levels. Again the slave fermion density fluctuations are absent even at the "zeroth" order of the mean field theory. The time reversal symmetry and the parity are spontaneously broken. After integrating out the slave fermions the effective lattice gauge theory contains a Chern-Simons term due to the T and P breaking. Because of the Chern-Simons term, the gauge field fluctuations have finite energy gap and can only mediate short range interactions. The slave fermions are not confined. The quasiparticles (spinons) are dressed slave fermions which carry spin $\frac{1}{2}$ and have fractional statistics. The mean field chiral spin state studied in Ref. 3,10,11,12 is a typical example of this possibility.

C) The third way to obtain a reliable mean field theory is to break the $SU(2)$ gauge symmetry. In this case the gauge fluctuations obtain a finite energy gap due to the Anderson-Higgs mechanism which solve the infrared problem. Notice that there is no gapless excitations in the Higgs phase. The slave fermion gas is again incompressible. The above possibility can be achieved by requiring the plaquette variable of the mean field solution

$$P_{ijk..l}\ =\ U_{ij}\ U_{jk}\ ...\ U_{li} \tag{17}$$

completely breaks the $SU(2)$ symmetry. In this case η_{ij} must be non-zero. To see how the gauge bosons obtain a finite energy gap (or a finite mass), let us note that the free energy of (9) in general contains the following gauge invariant term

$$F = \text{Tr}\ (P_{ij..k} e^{ia^l_{ii'}\tau^l} P^\dagger_{i'j'..k'} e^{ia^l_{i'i}\tau^l}) \tag{18}$$

In the continuum limits (18) reduce to the mass term for the a^l_μ field:

$$F \propto \int d^2x |\text{Tr}[P, a^l_\mu \tau^l]|^2 \tag{19}$$

where $P = P_{ij..k}$. If P does not commute with the $SU(2)$ transformation (i.e. $P \neq \pm 1$), the free energy (19) will generate mass terms for the $SU(2)$ gauge field fluctuations around the mean field solution. But notice that the mass term (19) can only break $SU(2)$ down to $U(1)$. The gauge symmetry that commute with P remain unbroken. *e.g.,* when $P \propto \tau^3$ we have $F \propto (a^1)^2 + (a^2)^2$. The gauge field a^3 remain gapless and the gauge symmetry generated by τ^3 remain unbroken. To break the $SU(2)$ gauge symmetry completely we need another plaquette variable $P' = P'_{ij..k}$ that do not commute with P. The total mass terms now become

$$F = \int d^2x |\text{Tr}[P, a^l_\mu \tau^l]|^2 + \int d^2x |\text{Tr}[P', a^l_\mu \tau^l]|^2 \Big] \tag{20}$$

We see that at least two mass terms are necessary to completely break the $SU(2)$ gauge symmetry. Once the $SU(2)$ gauge symmetry is broken, the infrared problems are well under control due to the finite mass term. In this case the low energy properties of the mean field theory and the corresponding spin liquid state can reliably derived.

In all the cases discussed above the infrared problem of the slave fermions with the $SU(2)$ gauge interaction is resolved by opening energy gaps. Due to those energy gaps the infrared behavior of the mean field theory is well under control. We can obtain the low energy properties of the mean field theory quite reliably. Because of the infrared stability of the theory, those low energy properties are expected to be robust, at least qualitatively, against small perturbations. Therefore it is reasonable to assume that the low energy properties of the mean field theory qualitatively describe the low energy properties of the spin liquid state. Because of this reason we will say the mean field theories satisfying A), B) or C) to be dynamically stable. The dynamically stable mean field theories can lead to a reliable description of the low energy properties of the spin liquid state. Many mean field theory studied before are not dynamically stable. It is thus dangerous to extract the low energy properties of the spin liquid state, *e.g.,* the quantum numbers of the quasiparticles, from those dynamically unstable mean field theories.

In the mean field theories discussed in A–C), the slave fermions form incompressible states. The dynamical stability of the mean field states and the incompressibility of the slave fermions are closely related. Given a slave fermion system with a gauge interaction, one may ask when the slave fermions are incompressible. There are only three well understood incompressible states for an fermion system. The fermions may form a band insulator which corresponds to the case A). If a “magnetic” field is dynamically generated the fermions may form an incompressible quantum Hall liquid which corresponds to the case B). The fermions can also form a superconducting state, which is again incompressible. Notice that a superconducting state, in contrast to a superfluid state, contains no gapless excitations. It certainly contains no gapless density fluctuations. This corresponds to the case C).

The spin liquid states correspond to the case B) have been studied in detail in Ref. 3. In the next section, as an example, we will study a T and P symmetric spin liquid state corresponding to the case C).

III. T AND P SYMMETRIC SPIN LIQUID STATE – MEAN FIELD THEORY OF THE S-RVB STATE.

Again we will consider the frustrated Hamiltonian (1). The T and P symmetric spin liquid state is given by the following mean field ansatz:

$$\begin{cases} \chi_{i,\ i+\hat{x}} = \chi_{i,\ i+\hat{y}} = \chi \\ \eta_{i,\ i+\hat{x}+\hat{y}} = \eta^{\dagger}_{i,i-\hat{x}+\hat{y}} = \eta \\ \chi_{ij} = \eta_{ij} = 0, \quad \text{otherwise}, \\ a_0^1 \neq 0, \quad a_0^2 = a_0^3 = 0 \end{cases} \tag{21}$$

where χ is a real parameter and η is a complex parameter. The corresponding $SU(2)$ link variables U_{ij} are given by:

$$\begin{aligned} U_{i,\ i+\hat{x}} = U_{i,\ i+\hat{y}} &= -\chi\tau^3 \\ U_{i,\ i+\hat{x}+\hat{y}} &= \mathrm{Re}(\eta)\tau^1 + \mathrm{Im}(\eta)\tau^2 \\ U_{i,\ i-\hat{x}+\hat{y}} &= \mathrm{Re}(\eta)\tau^1 - \mathrm{Im}(\eta)\tau^2 \end{aligned} \tag{22}$$

From the mean field ansatz we can easily obtain the following results:

1) The mean field ansatz is invariant under the translation and the spin rotation. Therefore, the mean field ansatz describes a spin liquid state. Or more precisely the ground state of H in (12), after the Gutzwiller projection, gives rise to a wave function of spin liquid state (with translation symmetry).

2) $U_{ij} = U_{ji}^{\dagger} = U_{ji}$. Thus the links are non-oriented.

3) Around triangles we have

$$\begin{aligned} U_{23}\ U_{31}\ U_{12} = U_{43}\ U_{31}\ U_{14} = -\chi^2(\mathrm{Re}(\eta)\tau^1 + \mathrm{Im}(\eta)\tau^2) \\ U_{12}\ U_{24}\ U_{41} = U_{32}\ U_{24}\ U_{43} = -\chi^2(\mathrm{Re}(\eta)\tau^1 - \mathrm{Im}(\eta)\tau^2) \end{aligned} \tag{23}$$

Because τ^1 and τ^2 do not commute, the $SU(2)$ gauge symmetry is completely broken if $\mathrm{Im}\eta \neq 0$. All gauge bosons obtain finite masses (or, energy gaps).[13]

4) The mean field ansatz is invariant under the parity $P:\ x \leftrightarrow y$. Therefore the spin liquid state is P invariant.

5) Later we will show that the constraint (11) can be completely satisfied by properly choosing the value of a_0^1.

However, the mean field ansatz is not invariant under 90° rotation and one might naively expect that the spin liquid state breaks the 90° rotation symmetry. But remember that U_{ij} are not physically observable. Only the $SU(2)$ gauge equivalent classes of U_{ij} are physical variables. Therefore the non-invariance of U_{ij} under the 90° rotation does not imply the non-invariance of the gauge equivalence classes of U_{ij}. In the following we will show that the gauge equivalent classes of the mean field ansatz (21) (or (22)) is invariant under the 90° rotation. Hence the corresponding spin liquid state respects the 90° rotation symmetry.

Under the 90° rotation $U_{ij} \to U'_{ij}$ where

$$\begin{aligned} U'_{i,\ i+\hat{x}} &= U'_{i,\ i+\hat{y}} = -\chi\tau^3 \\ U'_{i+\hat{x}+\hat{y}} &= \mathrm{Re}(\eta)\tau^1 - \mathrm{Im}(\eta)\tau^2 \\ U'_{i-\hat{x}+\hat{y}} &= \mathrm{Re}(\eta)\tau^1 + \mathrm{Im}(\eta)\tau^2 \end{aligned} \tag{24}$$

The two mean field ansatz U_{ij} and U'_{ij} can be shown to be gauge equivalent

$$U_{ij} = W_i U'_{ij} W_i^\dagger \tag{25}$$

where

$$W_i = (-)^i \tau^1 \tag{26}$$

The gauge transformation W_i also leave a_0^1 invariant. Therefore the mean field ansatz actually describe a spin liquid state which respect the 90° rotation symmetry.

Under T the mean field ansatz U_{ij} is changed to $U'_{ij} = U^*_{ij}$. Using the same gauge transformation (26) we can show that U_{ij} and U'_{ij} are gauge eqivalent. Thus our spin liquid state also respects time reversal symmetry.

Let $|\Phi\rangle$ be the ground state of the mean field Hamiltonian (12). We would like to view $|\Phi\rangle$ as a trial wave function. The mean field ground state energy is given by $E = \langle\Phi|H|\Phi\rangle$ where H is given by (3). Notice that for the mean field ansatz (21) the mean field Hamiltonian H_{mean} can be written as

$$\begin{aligned} \frac{H_{mean}}{J_1\chi} = &\sum_{n.n.} s^\dagger_{i\alpha}\ s_{j\alpha} + \sum_{n.n.n.} (\zeta s_{i\alpha}\ s_{j\beta}\ \epsilon^{\alpha\beta} + h.c.) \\ &+ \sum (\tilde{a}_0^1 s_{i\alpha} s_{i\beta} \epsilon_{\alpha\beta} + h.c.) \end{aligned} \tag{27}$$

where $\zeta = \frac{J_2\eta}{J_1\chi}$ and $\tilde{a}_0^1 = a_0^1/J_1\chi$. From (27) we see that $|\Phi\rangle$ only depend on ζ ($\tilde{a}_0^1$ is determined by (11)). Thus the mean field energy $E(\zeta)$ is a function of ζ.

When $J_2/J_1 = 0$ we find that $E(\zeta)$ is minimized at $\zeta = 0$. The mean field solution is given by $\chi = 0.41$ and $\eta = 0$. When $0 < J_2/J_1 < 1.4$, $E(\zeta)$ is minimized at a non-zero ζ with $\mathrm{Im}\zeta,\ \mathrm{Re}\zeta \neq 0$. Hence the mean field solution also satisfies $\mathrm{Im}\eta,\ \mathrm{Re}\eta \neq 0$ and the $SU(2)$ gauge symmetry is completely broken. If J_2/J_1 is not too small (*e.g.*, $J_2/J_1 > 0.3$) we have $\mathrm{Im}\eta/\mathrm{Re}\eta \sim 1$. When $J_2/J_1 > 1.4$ we find that $E(\zeta)$ is minimized at $|\zeta| = \infty$ and $\mathrm{Im}\zeta/\mathrm{Re}\zeta = 1$. In this case the mean field solution is given by $\chi = 0$ and $\eta = 0.48e^{i\pi/4}$.

We would like to emphasize that the self consistency of our mean field theory require $\eta \neq 0$ (only in this case can we break the $SU(2)$ gauge symmetry). From the above discussion we see that the frustration (*i.e.*, non-zero J_2) is crucial for the exsitance of our T and P symmetric gapful spin liquid state.

We would like to remark that although the mean field ansatz with $\mathrm{Im}\eta, \mathrm{Re}\eta \neq 0$ minimize the mean field ground state energy for $0 < J_2/J_1 < 1.4$, this does not imply that the spin Hamiltonian (1) support such a spin liquid state. This is because the quantum fluctuations are large and the mean field calculation is not reliable. The above calculation is just to demenstrate that the mean field ansatz is self consistent. In the following we will assume that the mean field solution given by (21) with $\mathrm{Im}\eta, \mathrm{Re}\eta \neq 0$ is stable. In this case the $SU(2)$ gauge symmetry is completely broken. If the simplest frustrated Heisenberg model (1) does not give rise to this result, we will assume such a result can by realized in some other spin models.

Assuming the stability of the mean field ground state, we can study the properties of the quasiparticle excitations. In the "zeroth" order mean field theory, the dynamics of the quasiparticles is described by the mean field Hamiltonian (12). Notice that the mean field ansatz actually describes a BCS "superconductor". We may directly apply the results in the mean field theory of the BCS superconductor to our case. We would like to point out that the "superconducting" order parameter in our case has a $s+id$ symmetry. The order parameter does not have a definite angular momentum. This is possible because the gap equation is non-linear. From (12) we see that "superconducting" gap is given by

$$\Delta_k = 2J_2\Big[\eta\cos(k_x+k_y) + \eta^\dagger\cos(-k_x+k_y)\Big] + a_0^1 + ia_0^2 \tag{28}$$

and the slave fermion spectrum is given by

$$\epsilon_k = 4J_1\chi(\cos k_x + \cos k_y) + a_0^3 \tag{29}$$

a_0^l are Lagrangian mutipliers which are chosen enforce the constraint

$$\chi_{ii} = 1, \quad \eta_{ii} = 0 \tag{30}$$

χ_{ij} and η_{ij} can be determined from Δ_k and ϵ_k:

$$\begin{aligned} \chi_k &= \sum_j \chi_{ij}\, e^{i\mathbf{k}\cdot(\mathbf{j}-\mathbf{i})} = \Big(1 - \frac{\epsilon_k}{E_k}\Big) \\ \eta_k &= \sum_j \eta_{ij}\, e^{i\mathbf{k}\cdot(\mathbf{j}-\mathbf{i})} = \frac{1}{2}\frac{\Delta_k}{E_k} \end{aligned} \tag{31}$$

where

$$E_k = \sqrt{\epsilon_k^2 + |\Delta_k|^2} \tag{32}$$

The constraint (30) reduces to

$$\begin{aligned} \chi_{i,\,i} &= \frac{1}{N}\sum_k \Big(1 - \frac{\epsilon_k}{E_k}\Big) \\ \eta_{i,\,i} &= \frac{1}{N}\sum_k \frac{1}{2}\frac{\Delta_k}{E_k} \end{aligned} \tag{33}$$

From (33) we find that the constraint (11) can be satisfied if we choose $a_0^3 = a_0^2 = 0$. But in general $a_0^1 \neq 0$.

From the BCS theory, we see that the spectrum of the quasi-particle excitations is given by E_k. The quasi-particles are fermions with $\frac{1}{2}$ spin and zero electrical charge. Notice that the quasi-particle excitations has finite energy gap, as one can see from (32).

We would like to stress that the mean field state studied above is *not* an (electrical) superconductor. It is only a "superconductor" for the dynamical (sometimes is called fictitious) gauge field a_μ^3 arisen from the phase fluctuations of χ_{ij}. For the electromagnetic field our mean field state is an insulator.

In the "first" order mean field theory the collective fluctuations of the phases of χ_{ij} and η_{ij} are included. From the previous discussion we see that these fluctuations correspond to an $SU(2)$ gauge field. Sometimes the $SU(2)$ gauge fluctuations may drastically change the structure of ground state. The quasi-particles in the "zeroth" mean field theory may disappear from the physical spectrum due to possible confinement of the $SU(2)$ gauge field. If this happens the "zeroth" order mean field results provide little information about the physical properties of the true spin liquid state.

However, in our mean field state given by (21), the $SU(2)$ gauge symmetry is broken. The gauge bosons obtain finite masses. The infrared behavior of the $SU(2)$ gauge fluctuations are well under control. Those fluctuations do not qualitatively change the structure of the "zeroth" order mean field ground state. The massive $SU(2)$ gauge fluctuations can only mediate short range interactions between quasi-particles in the "zeroth" order mean field theory, and there is no confinement. Therefore the quasi-particles in the "zeroth" order mean field theory also appears as the quasi-particles in the "first" order mean field theory. Those quasi-particles qualitatively describe the excitations in the corresponding spin liquid state. In particular the spin liquid state considered here support neutral spin $\frac{1}{2}$ fermionic excitations. These excitations are spinons. The energy spectrum of the spinons is qualitatively given by E_k and has finite energy gap.

To summarize we have described a mean field theory of a spin liquid state. The spin liquid state respects the translation symmetry and the 90° rotation symmetry. It also respects T and P. The spin excitations in the spin liquid state are found to be neutral spin $\frac{1}{2}$ spinons. The spinons obey Fermi statistics and have a finite energy gap. We would like to emphasize that the mean field theory discussed above is dynamically stable. Therefore, we expect the mean field theory qualitatively describes the properties of the actual spin liquid state supported by some spin Hamiltonian.

IV. THE DOPED S-RVB STATE.

In this section we are going to study t-J model in the low doping limit assuming that at zero doping the ground state of the spin system is the spin liquid state described by the mean field theory in the last section. Introducing the slave boson b, we may write the t-J model as:

$$\begin{aligned} H = & \sum J_{ij}\ s_{i\alpha}^\dagger\ s_{j\alpha} s_{j\beta}^\dagger s_{j\beta} \\ & + \sum t\ s_{i\alpha}^\dagger s_{j\alpha}\ b_j^\dagger b_i \end{aligned} \tag{34}$$

with the constraint:

$$s_{i\alpha}^\dagger s_{i\alpha}\ +\ b_i^\dagger b_i = 1 \tag{35}$$

on the state in the physical Hilbert space. Introducing the H-S field χ_{ij} and η_{ij} we obtain the mean field Hamiltonian at non-zero doping

$$
\begin{aligned}
H_{mean} = & \sum J_{ij} \left[\chi_{ij}\ s^\dagger_{i\alpha}\ s_{j\alpha}\ +\ \left(\eta_{ij}\ s^\dagger_{i\alpha}\ s^\dagger_{j\beta}\ \epsilon^{\alpha\beta}\ +\ h.c. \right) \right] \\
& + \sum t\chi_{ij} b^\dagger_j b_i + \sum a^3_0 (s^\dagger_{i\alpha} s_{i\alpha} + b^\dagger_i b_i - 1) \\
& + \sum [(a^1_0 + i a^2_0) s^\dagger_{i1} s^\dagger_{i2} + h.c.](1 - b^\dagger_i b_i)
\end{aligned}
\tag{36}
$$

At the "zeroth" order, the fluctuations of χ_{ij} and η_{ij} are ignored. From the above discussions we see that for the particular mean field state (21) the gauge fluctuations do not cause the infrared divergence and the "zeroth" order mean field theory gives a correct qualitative description of the spin liquid state. Therefore we expect (36) qualitatively describes the quasi-particle excitations in the doped spin liquid state.

In (36) b describes a charge e spinless boson which is called holon. At zeroth order, the hopping rate of the holons is given by $t\chi$. Because χ is order $O(1)$, the effective mass of the holon is of order $\frac{1}{t}$ which is close to the electron band mass. But in (36) we have ignore the spinon-holon interaction. After include the interaction we expect the effective mass of the holon to be much larger that $1/t$

The properties of the spinons and the holons have been studied by many people based on mean field theory and assuming the interaction between the spinons and the holons are weak.[14] However, many mean field theories studied before contain strong gauge field fluctuations which mediate a confining interaction between the spinons and the holons. In this paper we find a mean field spin liquid state in which the $SU(2)$ gauge field are massive and the interaction between the spinons and the holons is relatively weak. Thus many results of the spinons and the holons obtained in the previous studies may apply to our spin liquid state.

In addition to the spinons and the holons, there is another quasi-particle excitation in our spin liquid state. This excitation appears as topological soliton in the mean field theory. Notice that in our mean field state the $SU(2)$ gauge symmetry is broken by the Higgs fields in the adjoint representation of the $SU(2)$. In this case the $SU(2)$ gauge symmetry is broken down to Z_2 gauge symmetry. The quasi-particles in the Z_2 gauge theory are the Z_2-vortexes. These Z_2-vortexes are the new quasi-particle excitations mentioned above. In the mean field theory a Z_2-vortex is described by the following ansatz:

$$
\tilde{U}_{ij}\ = e^{-i\theta_i \tau^3/2}\ U_{ij}\ e^{i\theta_i\ \tau^3/2}
\tag{37}
$$

where θ_i is the angle of the ith site relative to the center of the Z_2-vortex (x, y):

$$
\mathrm{tg}\ \theta_i = \frac{i_x - x}{i_y - y}
\tag{38}
$$

Because $e^{i\theta\tau^3/2}|_{\ \theta=2\pi} = -1$, the holon (spinon) wave function will obtain a minus sign as a holon (spinon) moves around the Z_2-vortex. Therefore the Z_2-vortex behaves like a π flux vortex to the spinons and the holons.

Because of the Z_2-vortex, the $\frac{hc}{2e}$ flux have a finite energy even in the charge e holon condensed state.[15] In the superconducting state a Z_2-vortex and a bare $\frac{hc}{2e}$ magnetic vortex have infinite energy since the holon wave function change sign as a holon goes around the Z_2-vortex or the $\frac{hc}{2e}$ magnetic vortex. But the bound state of a Z_2-vortex and a $\frac{hc}{2e}$ magnetic vortex has a finite energy. The holon wave function does not change sign as a holon moves around such a bound state. Therefore even in the charge e holon condensed state, the minimum flux quantum is still $\frac{hc}{2e}$. The charge e holon superconducting state does not contradict with the experimental results. We also notice that binding a Z_2-vortex to a holon change the statistics of the holon from bosonic to fermionic. Similarly, the bound state of a spinon and a Z_2-vortex behave like a boson. This phenomenon has been discussed in detail in Ref. 16.

V. TOPOLOGICAL ORDERS IN THE SPIN LIQUID STATES

In this paper and in Ref. 3 the s-RVB state and the chiral spin state are constructed by using the mean field theory. Both spin liquid states have finite energy gap and respect the translation symmetry. At half filling such spin liquid states are Mott insulators, or more precisely, an insulator with *odd* number of electrons per unit cell.

However, the spin liquid states are constructed by using the mean field theory which involves unphysical fields (e.g. the slave fermion field s_i and slave boson field b_i). Our description and characterization of the spin liquid states are also in term of those unphysical quantities. It is very important to understand how to characterize the spin liquid states (or the Mott insulators) using physical properties. Especially we would like to know whether there is any physical order parameter which characterizes the spin liquid phase studied in this paper.

In this section we are going to argue that the spin liquid states studied here cannot be completely characterized by their symmetry properties and by the order parameters associated with broken symmetries. Our spin liquid states contain non-trivial topological orders and are characterized by those topological orders.

First let us discuss what is the topological orders.[17] Consider a rigid state containing no gapless quasiparticle excitations. Because of the finite energy gap such a system is almost trivial at low energies. The only non-trivial feature at low energies comes from degenerate ground states. One may naively think the degenerate ground states always come from broken discrete symmetries and conclude that a rigid state is characterized by its symmetry properties. However this naive expectation is not correct. It has been shown that the fractional quantum Hall (FQH) fluid support degenerate ground state even when the Hamiltonian contains no discrete symmetries.[18] The ground state degeneracy is shown to be robust against arbitrary perturbations despite there is no symmetry to protect it. Furthermore the number of the degenerate ground depend on the topology of the space. All of these results point to one thing: rigid systems, such as FQH states, may have non-trivial infrared fixed points which cannot be characterized by broken symmetries. Those unusual infrared fixed points are said to be characterized by topological orders.

Both the chiral spin state and the s-RVB state are rigid states. In Ref. 19,17 the chiral spin states are shown to have non-trivial topological orders. In this section we will show that the s-RVB state also contains non-trivial topological orders.

As in the FQH states, the non-trivial topological orders can be detected by measuring the ground state degeneracy of the system on compactified spaces. The ground state degeneracy and its relation to the Z_2 vortex has been studied in Ref. 16 within the nearest neighbor dimer model.[4] It was shown that the s-RVB state has 2^{2g} degenerate ground states on a genus g Riemann surfaces. In the following we will describe the above result in terms of our mean field theory.

In the mean field theory the 2^{2g} degenerate ground states can be constructed by adding zero or one unit of the Z_2-flux through the $2g$ non-contractable loops on the genus g Riemann surface. On the torus the mean field ansatz for the four degenerate ground states is given by:

$$U_{ij}^{(m,\, n)} = e^{-i(m\frac{ix}{Lx}+n\frac{iy}{Ly})\tau^3\pi}\ U_{ij}e^{i(m\frac{ix}{Lx}+n\frac{iy}{Ly})\tau^3\pi} \tag{39}$$

where $m, n = 0,\ 1$ and L_x and L_y are the size of torus in the x and y directions. $U_{ij}^{(m,n)}$ with different m and n are *locally* gauge equivalent. Therefore, the free energies for different ansatz are expected to be the same (in the thermodynamical limit). Thus $U_{ij}^{(m,n)}$ describes the degenerate ground states of the system. However $U_{ij}^{(m,n)}$ with different m and n are not gauge equivalent in the global sense. A spinon propagating all the way around the tour in the $x(y)$ direction obtain a phase $e^{im\pi}$ ($e^{in\pi}$). Therefore $U_{ij}^{(mn)}$ describes different ground states. In other words the ground state wave functions for different m and n are orthogonal to each other.

On a finite compactified lattice, the only way for the system to tunnel from one ground state to another is through the following tunneling process. At first a pair of the Z_2-vortexes is created. One of the Z_2-vortexes propagates all the way around the torus and then annihilates with the other Z_2-vortex. Such a process effectively add a unit of the Z_2 flux to the hole of the torus and change m or n by 1. The different ground states can not tunnel into each other through any local fluctuations. As a direct consequence of this result, the energy split between different ground state on a finite lattice is expected to be of order $e^{-L/\xi}$.

In the mean field theory the degeneracy of ground states is a consequence of the gauge symmetry. The gauge symmetry remains to be exact even after we include an arbitrary perturbation to the original spin Hamiltonian:

$$\delta H = \Sigma\ \delta J_{ij}\ \mathbf{S}_i\ \cdot\ \mathbf{S}_j\ +\ ... \tag{40}$$

δH may break translation symmetry, rotation symmetry, *etc.* . The above arguments are still valid even for the modified Hamiltonian. The mean field ground states remain to be four fold degenerate. We expect this result remains to be true even beyond the mean field theory. The ground state degeneracy of our spin liquid state can not be changed by any perturbation as long as the perturbation is weak enough not to drive a phase transition. Therefore, the ground state degeneracy can be regarded as a quantum number characterizing the spin liquid state. A more complete characterization of the spin liquid can be obtained by studying the non-Abelian Barry's phase associated with the twisted Hamiltonians.[17]

From the above discussions we conclude that the ground state degeneracy of the s-RVB state is not due to broken discrete symmetries. This is because 1) The ground state degeneracy depends on the topology of the compactified space; 2) the ground state degeneracy is robust against arbitrary perturbations, even those perturbations which break all the symmetries in the Hamiltonian. 3) The energy split of the ground states is of order

$e^{-L/\xi}$ for a finite system of size L. If the ground state degeneracy was due to broken discrete symmetry the energy split would be at most of order $E^{-L^2/\ \xi^2}$. This result suggest that the s-RVB state studied in section 3 contains non-trivial topological orders.

The above discussion is essentialy an apllication of the standard Z_2 gauge theory to our mean field state. We hope to clarify the following points through the discussion in this section. A) The stability of the Z_2 vortex is connected to the breaking of the $SU(2)$ gauge symmetry. We know that the stability of the Z_2 vortex or the existance of the Z_2 gauge structure is crutial for the stability of the ground state degeneracy. It was shown that the dimer fluctuations in the nearest neighbor RVB state mediate a confining interaction between holons.[20] This indecates that the Z_2 vortex in the nearest neighbor RVB state is unstable. Therefore it is not clear whether the nearest neighbor RVB state is a generic state or not. What we have learnt from the above discussions is that the stability of the s-RVB state is insured by the gauge symmetry breaking. Our results suggest that the inclusion of the longer bonds in the dimer model may help to stablize the Z_2 vortex and to make the s-RVB state a generic state. B) The ground state degeneracy is due to the gauge symmetry in the mean field theory. The degeneracy is robust against any small perturbations.

The topological order is a very useful concept. Let us consider the following question: what is the difference between the spin-Peierls state and the s-RVB state? One may immediately say that the two states have different symmetries. But if we modify our Hamiltonians to break the translation and the rotation symmetries, then the two states will have the same symmetries. In this case we can still ask whether the two states are the same or not in the sense whether one state can be continuously deform into the other without phase transitions. When the translation and the rotation symmetries are broken, the spin-Peierls state only support a non degenerate ground state. While the s-RVB state still has four degenerate ground states on the torus. Therefore the spin-Peierls state and the s-RVB state are different even when they have the same symmetries. The two states differ by having different topological orders.

VI. DISCUSSIONS

In this paper we studied some mean field theories of spin liquid state. In those mean field theories the charge excitations have finite gap. Because the system is already incompressible at mean field level, the Gutzwiller projection does not drastically change the correlations in the field theories. Many properties of the spin liquid states can be obtained from the mean field theories. In particular for the cases B) and C) discussed in section 2, the gauge fluctuations in the mean field theories have finite gap and do not mediate any confining interactions. In those cases, the quasi-particle in the mean field theories directly correspond to the quasi-particles in the spin liquid states. Such mean field theories qualitatively describe all the properties of the spin liquid state and are said to be dynamically stable. The chiral spin state is one of the dynamically stable mean field states. The gauge bosons are massive due to the Chern-Simons term. In this paper we propose another dynamically stable mean field state which respect the T and P. The gauge bosons obtain finite masses from the Higgs mechanism. The quasi-particles in the corresponding spin liquid state are found to be the spinons with Fermi statistics and the holons with Bose statistics. Such a state corresponds to the s-RVB state. The s-RVB state is also shown to contain non-trivial topological orders. We would like to stress that the results obtained in this paper are not limited in two dimensions. They apply equally well to higher dimensions.

Many properties of our mean field state are closely related to those obtained in the dimer model.[4,16] However, in the simplest dimer model which contain only bonds connecting nearest neighbors, the dimer fluctuations are shown to induce a confining interaction between the holons and the spinons.[20] Also in this dimer model, the holons and the spinons on the even sublattice do not mix with the holons and the spinons on the odd sublattice. The studies in this paper suggest that the confining interactions in the dimer model may be removed by including the dimers connecting the next nearest neighbors. Those long dimers may also help to prevent the formation of the spin-Piererls state.

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