Chiral Luttinger Liquid and the Edge Excitations in the Fractional Quantum Hall States*

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ABSTRACT: The low energy effective theory of the edge excitations in the Fractional Quantum Hall (FQH) states is proposed. The edge excitations are shown to form a new kind of state which is called the Chiral Luttinger Liquid (χLL). The effective theory is exactly soluble. This enables us to easily calculate all the low energy properties of the edge excitations. We calculate the electron propagator and the spectral function, which clearly demonstrate the non-Fermi liquid behaviors of the χLL. We also calculate the interference effects between excitations on different edges. We demonstrate that the properties of the edge excitations are closely related to the properties of the FQH states on compactified spaces. Thus the properties of the edge excitations can be used to characterize the topological orders in the FQH states. We also show that the FQH states with filling fractions $\nu \neq \frac{1}{l}$ must have at least two branches of edge excitations.

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I. INTRODUCTION

In the past few years many people studied the low energy dynamical properties of the Quantum Hall (QH) states.[1] The experiments clearly observed gapless excitations in finite QH systems. It is generally believed that the gapless excitations are localized at the edges of the systems. This is because the QH states are incompressible and contain no bulk gapless excitations. Using Laughlin's arguments,[2] one can easily prove the existance of the gapless excitations in a finite QH system. The real non-trivial issue is to understand the dynamics of the edge excitations. For integral QH states,[3] the dynamical properties of the edge excitations are shown to be described by 1D Fermi liquid theory. While for the FQH states,[4] they are described by the $U(1)$ Kac-Moody (K-M) algebras. Some static properties (*e.g.,* DC transport properties) of the edge states in the FQH regime are studied in Ref. 5.

In general the gapless edge excitations may have many branches.[3,4,5] The dynamics of the edge excitations is generally described by several $U(1)$ K-M algebras in the *low energy* limit.[4] This is equivalent to say that the charge zero sector of the edge excitations is described by the charge zero sector of a Fermi liquid theory (in the low energy limit). Such a Fermi liquid theory contain many branches of fermions. The charges of the fermions are shown to satisfy a sum role[4]

$$\sum_I \frac{v_I}{|v_I|} q_I^2 = \nu e^2 \tag{1.1}$$

In (1.1) v_I and q_I are the velocities and the charges of the fermions in the Ith branch and ν is the filling fraction of the FQH state. In general the charges q_I can be irrational numbers.[6] The relation between the edge excitations and the Fermi liquid can be used to calculate many properties of the edge excitations. The responses of the edge states to external electromagnetic fields are calculated in Ref. 6, which lead to a practical way to experimentally measure the charges q_I carried the fermions.

However, as emphasized in Ref. 4 and Ref. 6, although the charge zero sector of the edge excitations are described by a Fermi liquid theory, the charged edge states may *not* be described by Fermi liquid theories. Especially the charges of the charged edge states may not be multiples of q_I. Therefore, to be accurate, we will call q_I the optical charges of the edge excitations. This is because q_I are measured only through the current correlation functions and do not correspond to the charges of the charged edge states. Strictly speaking the edge states in the FQH regime are not Fermi liquids. In this paper we will derive an effective theory which describes both the charged and the neutral excited edge states. We will concentrate on the non-Fermi liquid behaviors of the edge excitations. Although the edge states are not Fermi liquids, the effective theory of the edge excitations is still exactly soluble. One can easily obtain all the low energy properties of the edge excitations from the effective theory.

II. THE EDGE EXCITATIONS ON A DISC

Consider a FQH state on a disc with filling fraction ν. Let us assume that the edge excitations have only one branch. This implies that the charge zero sector of the edge excitations are described by a single $U(1)$ K-M algebra:[4]

$$\begin{aligned}
[j^+_{k'}, j^+_k] &= e^2 \frac{\nu}{2\pi} k \delta_{k+k'} \\
[j^+_{k'}, j^-_k] &= [j^-_{k'}, j^-_k] = 0 \\
[H, j^\pm_k] &= vkj^\pm_k
\end{aligned} \tag{2.1}$$

where $j^\pm_I = \frac{1}{2}\left(j^0 \pm \frac{1}{v} j^\sigma\right)$, $j^\alpha_k = \int d\sigma \frac{1}{\sqrt{L}} e^{i\sigma k} j^\alpha(\sigma)$ and L is the length of the edge. The (optical) charge of the fermions in the corresponding Fermi liquid theory is given by $q = \sqrt{\nu} = 1/\sqrt{n}$.

The charged excited states arise from adding (or subtracting) electrons to (from) the edge. Therefore those charged states are generated by electron creation (or annihilation) operators $\psi^\dagger$ (or ψ). The electron operator $\psi^\dagger(\sigma)$ create an unit charge which is localized at σ. Therefore we have

$$[j^0(\sigma'), \psi^\dagger(\sigma)] = e\psi^\dagger(\sigma)\delta(\sigma - \sigma') \tag{2.2}$$

The total charge operater is given by $Q = \int d\sigma e^{i\sigma k} j^0(\sigma)$. Because all the low lying excitations have the same velocity v, ψ also satisfies

$$[H, \psi_k] = vk\psi_k \tag{2.3}$$

If ψ_k had a velocity different from v, the current operator $j^\sigma_k = \sum \psi_{k'}(k+k')\psi_{k+k'}$ would also have that velocity. This would contradict with (2.1).

The total Hilbert space of the edge excitations is generated not only by the current operator j^+ but also by the charged operator ψ. Therefore the Hilbert space of the edge excitations forms a representation of the algebra (2.1–2.3). To understand the properties of the charged edge states, we first need to find the representation of the algebra (2.1–2.3).

The structure of the Hilbert space of the edge excitations can be understood even without any calculations. First, the charge zero sector of the edge states form an irreducible representation of the $U(1)$ K-M algebra. The Hilbert space of such a representation is denoted as $\mathcal{H}_{KM}$. The charge e excited states are obtained by adding an electron to the system. The system with one more electron is essentially identical to the original system. Thus the charge e sector also form the irreducible representation of the K-M algebra. A similar result can be obtained for a general charge Ie sector. From the above discussions, we see that the total Hilbert space of the edge excitations is given by

$$\mathcal{H}_{disc} = \bigoplus_I \mathcal{H}^{(I)}_{KM} = \mathcal{H}_{KM} \otimes \mathcal{H}_p$$

where $\mathcal{H}_p$ is spanned by the states $|I\rangle$. The state $|I\rangle$ has charge Ie and $\mathcal{H}^{(I)}_{KM} = \mathcal{H}_{KM} \otimes \{|I\rangle\}$ corresponds to the charge Ie sector.

In the following we are going to show that the representation of the algebra (2.1–2.3) can be constructed from chiral boson theories. For convenience we will assume that the disc has a unit radius (*i.e.*, $L = 2\pi$) and set $e = v = 1$. Chiral boson theory is defined by the Lagrangian[7]

$$\mathcal{L} = \frac{1}{8\pi}\left[(\partial_0\phi)^2 - (\partial_\sigma\phi)^2\right] \tag{2.4}$$

where the real scalar field ϕ satisfies the "chiral" constraint

$$(\partial_0 - \partial_\sigma)\phi = 0 \tag{2.5}$$

In the following, we will follow Ref. 8 to quantize the chiral boson theory (2.4–2.5). The operator ϕ satisfies the equation of motion

$$(\partial_0 - \partial_\sigma)(\partial_0 + \partial_\sigma)\phi = 0 \tag{2.7}$$

The solutions of (2.6) take the form

$$\phi = \phi_0 + \tilde{\phi}_0 + p_\phi(t+\sigma) + \tilde{p}_\phi(t-\sigma) + i\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n e^{-in(t+\sigma)} + \tilde{\alpha}_n e^{-in(t-\sigma)}\right) \tag{2.7}$$

The canonical momentum of ϕ is given by $\pi = \frac{1}{4\pi}\partial_0\phi$:

$$\pi = \frac{1}{4\pi}(p_\phi + \tilde{p}_\phi) + \frac{1}{4\pi}\sum_{n\neq 0}\left(\alpha_n e^{-in(t+\sigma)} + \tilde{\alpha}_n e^{-in(t-\sigma)}\right) \tag{2.8}$$

p_ϕ and α_n describe the left moving excitations while $\tilde{p}_\phi$ and $\tilde{\alpha}_n$ describe the right moving ones. From the commutator between ϕ and π we find that $\tilde{\phi}_0$, $\tilde{p}_\phi$ and $\tilde{\alpha}_n$ satisfy the algebra

$$\begin{aligned}[][\tilde{\alpha}_n, \tilde{\alpha}_m] =& n\delta_{n+m} \\ [\tilde{\phi}_0, \tilde{p}_\phi] =& i \\ \text{others} =& 0.\end{aligned} \tag{2.9}$$

and ϕ_0, p_ϕ and α_n satisfy

$$\begin{aligned}[][\alpha_n, \alpha_m] =& n\delta_{n+m} \\ [\phi_0, p_\phi] =& i \\ \text{others} =& 0.\end{aligned} \tag{2.10}$$

At this stage we may impose the constraint (2.5) by dropping $\tilde{\phi}_0$, $\tilde{p}_\phi$ and $\tilde{\alpha}_n$. A more systematic and careful treatment of the chiral boson theory can be found in Ref. 7. Notice that algebra (2.10) just describe many independent oscillators. The Hilbert space of the chiral boson theory is defined as the Fock space of the oscillator algebra (2.10). The operators α_n generate the irreducible representation of the K-M algebra $\mathcal{H}_{KM}$. The space generated by the "zero modes" ϕ_0 and p_ϕ needs more careful treatment and will be discussed later. The Hamiltonian of the chiral boson theory is given by

$$H = \frac{1}{2}p_\phi^2 + \sum_{n>0}\alpha_n\alpha_{-n} \tag{2.11}$$

The electrical current in the chiral boson theory is identified as

$$j^\alpha = \frac{\sqrt{\nu}}{2\pi}\epsilon^{\alpha\beta}\partial_\beta\phi_L \tag{2.12}$$

where

$$\phi_L(t,\sigma) = \phi_0 + p_\phi(t+\sigma) + i\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in(t+\sigma)} \tag{2.13}$$

The total charge operator is

$$Q = \sqrt{\nu}p_\phi = -\sqrt{\nu}i\partial_{\phi_0} \tag{2.14}$$

Using (2.10) one can explicitly check that the current in (2.12) satisfy the K-M algebra (2.1). Therefore the Hilbert space of the chiral boson theory form a representation of the K-M algebra (2.1).

The charged operators in the chiral boson theory have a form $\Psi(\sigma) =: e^{i\gamma\phi_L(\sigma)} :$. The operater $\Psi(\sigma)$ create a localized charge at σ. Because we want to identify the charged operator Ψ as an electron operator, Ψ must satisfy anti-commutation relation:

$$\Psi(\sigma)\Psi(\sigma') = -\Psi(\sigma')\Psi(\sigma), \qquad \sigma' \neq \sigma \tag{2.15}$$

Using the formula

$$e^A e^B = e^{[A,B]} e^B e^A \tag{2.16}$$

we find that[9]

$$\Psi(\sigma)\Psi(\sigma') = e^{i\gamma^2(\sigma+\sigma')/2}(e^{-i\sigma} - e^{-i\sigma'})^{\gamma^2} : e^{i\gamma[\phi_L(\sigma)+\phi_L(\sigma')]} : \tag{2.17}$$

Therefore Ψ is a fermionic operator if $\gamma^2 \equiv l$ is an odd integer.

In order to identify the operator Ψ as an electron operator, we not only require Ψ to be a fermionic operator, we also require Ψ to carry unit charge. From (2.14) we see that the charge of Ψ is given by $\gamma\sqrt{\nu}$. This implies that only when the filling fraction satisfies $\nu = \frac{1}{l}$, can the operator Ψ be identified as an electron operator:

$$\psi^\dagger(t,\sigma) = \eta\Psi = \eta e^{i\sqrt{l}\phi_L(t,\sigma)} \tag{2.18}$$

where η is a constant which may depend on the cut-off. One can easily check that ψ satisfies (2.2). As a direct consequence of the above result, a FQH state with filling fraction $\nu \neq 1/l$ must have more than one branch of edge excitations.

The physical Hilbert space of the chiral boson theory (2.4–2.5) is generated by the operators j_k^+ and ψ, or equivalently by α_n and $e^{i\sqrt{l}\phi_0}$. The operator $e^{i\sqrt{l}\phi_0}$ generates the charged exited states. Because α_n and $e^{i\sqrt{l}\phi_0}$ commute, the Hilbert space can be written as

$$\mathcal{H}_{KM} \otimes \mathcal{H}_p \tag{2.19}$$

where $\mathcal{H}_p$ is spanned by the states $|I\rangle$. The states $|I\rangle$ carries a charge $Q = Ie$ and $p_\phi = I\sqrt{l}$. Thus the Hilbert space of the chiral boson theory is identical to the Hilbert space $\mathcal{H}_{disc}$ that we obtained before.

The commutation relation (2.3) can be easily derived. First, notice that

$$[H, \psi(t,\sigma)] = -i\partial_0\psi(t,\sigma) \tag{2.20}$$

From (2.13) we see that $\psi(t,\sigma)$ depend on t and σ only through the combination $t+\sigma$. This implies that

$$\partial_0\psi(t,\sigma)=\partial_\sigma\psi(t,\sigma) \tag{2.21}$$

(2.3) can be easily obtained from (2.20) and (2.21). We find that the chiral boson theory (2.4–2.5) together with the quantization condition

$$p_\phi=\sqrt{l}\times\text{integers} \tag{2.22}$$

form a representation of the algebra (2.1–2.3).

Let us summarize our results. Consider a FQH state on a disc with a filling fraction ν. Assume that the edge excitations only have one branch and assume that it cost infinite small energy to add a single electron to the FQH state. Under those assumptions we show that the edge excitations of such a FQH state are described by the chiral boson theory (2.4–2.5) and (2.22). The chiral boson theory contains charge e fermion operator only when $\nu=\frac{1}{l}$ where l is an odd integer. Such an operator is identified as the electron operator. Therefore, a FQH state with filling fraction $\nu\neq 1/l$ must have more than one branch of edge excitations. For the $\nu=1/l$ FQH states, Haldane suggested that the edge excitations only have one branch.[10] In this case the edge excitations are described by the chiral boson theory (2.4–2.5) and (2.22). The Hilbert space of the chiral boson theory is generated by the operators α_n and ψ.

The electron Green function can be calculated using (2.16). We find that

$$\left\langle\psi^\dagger(t,\sigma)\psi(0,0)\right\rangle=\eta^2e^{-il(t+\sigma)/2}(e^{-i(t+\sigma)}-1)^{-l} \tag{2.23}$$

In the thermodynamic limit, $\sigma\ll L=2\pi$ and $t\ll L/v=2\pi$, we have

$$\left\langle\psi^\dagger(t,\sigma)\psi(0,0)\right\rangle=\eta^2\left(\frac{i}{t+\sigma}\right)^l \tag{2.24}$$

In the momentum space (2.24) becomes

$$\left\langle\psi_k^\dagger\psi_k\right\rangle\propto\frac{(\omega-k)^{l-1}}{\omega+k-i\delta}. \tag{2.25}$$

If electrons are described by Fermi liquid theory the Green function should be

$$\left\langle\psi^\dagger(t,\sigma)\psi(0,0)\right\rangle=\frac{i}{t+\sigma} \tag{2.26}$$

The anomalous exponent in the propagator (2.24) implies that the electrons on the edge of the FQH states do not form a Fermi liquid. The electrons are strongly correlated and form a new kind of state. Those states resemble the Luttinger liquid[11] in which the electron propagator also has an anomalous exponent. Because the excitations in our states move only in one direction, we will call such states Chiral Luttinger Liquids (χLL).

The Luttinger liquids contain both right moving and left moving excitations, while the χLL contain only left (or right) moving excitations. Because the chiral property of the χLL, the exponent in the electron propagator is expected to be a topological invariant. In the chiral boson theory considered here, the exponent is given by $\frac{1}{\nu}$ which is quantized as an odd integer. The exponent remain unchanged no matter how we perturb the Hamiltonian. In contrast, the exponent of the electron propagator in the Luttinger liquid can take arbitrary real values. The exponent depend on the interactions between electrons and is not a topological invariant. From the above discussion we see that the χLL and the Luttinger liquid have some fundamental distinctions.

However as pointed out in Ref. 4 and Ref. 6, the edge states of FQH systems are closely related to a Fermi liquid of *charge* $q = \frac{1}{\sqrt{l}}$ *fermions*. Or more precisely, the charge zero sector of the χLL is described by the charge zero sector of the charge $q = \frac{1}{\sqrt{l}}$ Fermi liquid theory. If we only interested in the process that conserve the total charge of the system, the χLL can be regarded as a Fermi liquid. But the charged excited states are not described by the charge q Fermi liquid theory. In particular the charges of the edge states are quantized as integers instead of as multiples of $\frac{1}{\sqrt{l}}$.

As we increase the size of the system, the constraint on the total charge becomes less and less important. In the fact the χLL and the charge q Fermi liquid have the same thermodynamic properties. We may effectively treat the edge states as a charge $q = \frac{1}{\sqrt{l}}$ Fermi liquid if we only interested in those thermodynamic properties.

To further demonstrate the similarity between the χLL and the Fermi Liquid, we would like to calculate the "edge capacitance" of the FQH states. Assume that a $\nu = 1/l$ FQH state is in equilibrium with a charge reservoir of voltage V. The total charge of the FQH state is a function of V, $Q = Q(V)$. The edge capacitance is defined as

$$C = \frac{dQ}{dV} \tag{2.27}$$

The edge capacitance can also be obtained from

$$\frac{1}{C} = \frac{d^2E(Q)}{d^2Q} \tag{2.28}$$

where $E(Q)$ is the total energy of the FQH state. Comparing (2.28) with (2.11) and (2.14), we find that the capacitance of the χLL is given by

$$C = \nu = \frac{1}{l}\frac{e^2L}{2\pi v}. \tag{2.29}$$

The capacitance of a charge q Fermi liquid is given by q^2N_0 where $N_0 = 1$ is the density of states of the Fermi liquid. We find that (2.29) is also the capacitance of the charge $q = \frac{1}{\sqrt{l}}$ Fermi liquid. Therefore despite the charge quantization conditions are different, the capacitance of the χLL and the capacitance of the charge $q = \frac{1}{\sqrt{l}}$ Fermi liquid are identical.

To observe non-Fermi liquid behaviors of the χLL we need to use the processes which change the total charge of the system and probe the microscopic structures in the states. Electron tunneling and the photoemission and two such experiments. Those experiments measure the electron spectral function

$$n_{\omega,k} = \sum_n |\langle n|\psi_k|0\rangle|^2\delta(\omega - \omega_n) \tag{2.30}$$

where ω_n is the energy of the state $|n\rangle$. From the electron propagator we find that the spectral function and the electron "density of states" in the χLL are given by

$$\begin{aligned} n_{\omega,k} &\propto \omega^{l-1}\delta(\omega+k)\theta(-\omega) \\ N(\omega) &= \int \frac{dk}{2\pi} n_{\omega,k} \propto \omega^{l-1}\theta(-\omega) \end{aligned} \tag{2.31}$$

Measuring the spectral function allows us to determine the anomalous exponent.

The χLL are characterized by the following properties:

(1) The χLL contain a conserved current which form a $U(1)$ K-M algebra with a central charge $\frac{q^2}{2\pi}$. q is called the optical charge of the χLL.

(2) The χLL contain a local charged operator. The $U(1)$ charge of the operator is given by q_0, which may not be equal to q.

(3) All the excited states have the same velocity $v = \frac{\epsilon}{k}$ where ϵ and k are total energy and the total momentum of the excited states.

When $q = q_0$ the χLL is just the a charge q Fermi liquid. When $q \neq q_0$ the χLL is different from Fermi liquid. But the charge zero sector of the χLL is still described by the charge zero sector of the charge q Fermi liquid.

Before ending this section we would like to discuss the relation between the chiral boson theory and the microscopic theory of the FQH states.[10] Consider a FQH system confined in a circular potential well. The filling fraction of the FQH states is $\nu = 1/l$. The ground state has an angular momentum M_0 and is given by the Laughlin wave function

$$\Phi_0(z_i) = \left[\prod_{i<j}(z_i - z_j)^l\right] e^{-\frac{1}{4}\sum_i |z_i|^2} \tag{2.32}$$

For such a simple FQH states, we may assume that the total energy of the system is a single valued smooth function of the total angular momentum:

$$E = E(M). \tag{2.33}$$

Haldane[10] pointed out that the charge zero edge excitations in such a system are generated by multiplying a symmetric polynomial to the ground state wave function:

$$|n_1, n_2, \ldots\rangle = P_{(n_1,n_2,\ldots)}(z_i)\Phi_0(z_i) \tag{2.34}$$

where

$$P_{(n_1,n_2,\ldots)}(z_i) = \sum_{\{i_1,\ldots,i_{n_1};j_1,\ldots\}} (z_{i_1}\ldots z_{i_{n_1}})(z_{j_1}^2 \ldots z_{j_{n_2}}^2)\ldots \tag{2.35}$$

The excited state $|n_1, n_2, \ldots\rangle$ has angular momentum $K + M_0$ and energy $K\frac{dE}{dM}$ (assume $E(M_0) = 0$) where $K = \sum jn_j$. Such a state is call the Kth level excited state. The number of the states at the Kth level is given by

$$N_K = \sum_{n_1,n_2,\ldots} \delta(\sum jn_j - K) \tag{2.36}$$

In the chiral boson theory the excited states is generated by the operators $\alpha_n,\ n>0$:

$$|n_1,n_2,\ldots\rangle=\alpha_1^{n_1}\alpha_2^{n_2}\ldots|0\rangle \tag{2.37}$$

The energy of the state $|n_1,n_2,\ldots\rangle$ is given by $Kv\frac{2\pi}{L}$ where $K=\sum jn_j$. We will again call such a state the Kth level state. The number of the Kth level states in the chiral boson theory is given by the same formula (2.36). Because $\frac{dE}{dM}$ is the angular velocity of the edge excitations $\frac{dE}{dM}=v\frac{2\pi}{L}$, the Kth level states in the FQH states and the Kth level states in the chiral boson theory have the same energy. Adding m electrons to the system increase the angular momentum by $lm(m+1)/2$ and the energy by $\Delta E=\frac{m(m+1)}{2}l\frac{dE}{dM}+m\mu$, where μ is the electron chemical potential. If we choose $\mu=-\frac{l}{2}\frac{dE}{dM}$ we find that $\Delta E=\frac{m^2}{2}lv\frac{2\pi}{L}$. This again agrees with the result in the chiral boson theory that the operator $e^{im\sqrt{l}\phi_0}$ create m electrons and increase the total energy by $\frac{1}{2}m^2lv\frac{2\pi}{L}$ (See (2.11)). From the above discussions we find that the microscopic FQH theory and the chiral boson theory give rise to the same Hilbert space and the same Hamiltonian for the low lying edge excitations. In this way we show that the chiral boson theory (2.4–2.5) and (2.22) describe all the dynamical properties of the edge excitations in the $\nu=\frac{1}{l}$ FQH states.

We would like to remark that comparing to the microscopic theory discussed above, the chiral boson theory is more general. The chiral boson theory remain to be valid even when the edge potential is not a smooth function and when the electron interaction is modified near the edge. This is because the K-M algebra (2.1) is a consequence of the gauge symmetry. The validity of the K-M algebra is independent of the detail structures of the edge configuration. The chiral boson theory also apply to the hierarchy FQH states which have many branches of edge excitations. The chiral boson theory in this case contain several boson fields, one boson field for each branch. For the $\nu\neq\frac{1}{l}$ FQH states, the total energy is not a smooth single valued function of M. In this case the symmetric polynomials do not generate all the low lying excitations.

III. EDGE EXCITATIONS ON A CYLINDER

In this section we will discuss the edge excitations on cylinder. We will assume the FQH state on the cylinder to have a filling fraction $\nu=1/l$. Since the cylinder has two edges, one may naively expect that the Hilbert space of the edge excitations on the cylinder is a direct product of the Hilbert spaces on the edges of two discs, $\mathcal{H}_{disc}\otimes\mathcal{H}_{disc}$, where $\mathcal{H}_{disc}=\mathcal{H}_{KM}\otimes\mathcal{H}_p$ is constructed in the last section. However this naive expectation is incorrect. There is a new kind of excitations on the edges of the cylinder. Those excitations are not contained in $\mathcal{H}_{disc}\otimes\mathcal{H}_{disc}$. Such new excitations transfer multiples of a *fractional* charge e/l from one edge to the other. The new excitations can be induced by adiabatic turning on unit flux going through the cylinder (Fig. 1). This adiabatic operation transfer e/l charge from one edge to the other. Because the charges in $\mathcal{H}_{disc}\otimes\mathcal{H}_{disc}$ is quantized as integers, such a excited state is not in $\mathcal{H}_{disc}\otimes\mathcal{H}_{disc}$. Turning on l unit flux transfer one electron between edges. This excitation adds an electron to one edge and subtracts an electron from the other edge. Such a excitation lies within the Hilbert space $\mathcal{H}_{disc}\otimes\mathcal{H}_{disc}$. From the above considerations we conclude that the edge excitations on the cylinder contain l sectors. Each sector is given by $\mathcal{H}_{disc}\otimes\mathcal{H}_{disc}$, since different sectors are related by adiabatic turning on unit flux. The total Hilbert space of the edge excitations on the cylinder is given by

$$\mathcal{H}_{cyl} = \bigoplus_{M=1}^{l} (\mathcal{H}_{disc} \otimes \mathcal{H}_{disc})^{(M)} = \mathcal{H}_{disc} \otimes \mathcal{H}_{disc} \otimes \mathcal{H}_{glo} \tag{3.1}$$

where $\mathcal{H}_{glo}$ contains l states $|M\rangle,\ M = 1, \dots, l$. Those states are generated by a operator T which transfers e/l charge from one edge to the other:

$$|M+1\rangle = T^M |0\rangle \tag{3.2}$$

The operator T is induced by adiabatic turning on unit flux. Such an operation induces transitions between sectors. But turning on l unit flux does not change the sectors. From the above discussions, we also see that the allowed values of the charges on the two edges are labeled by three integers I_1, I_2 and $M = 1, \dots, l$:

$$Q^{(R)} = (I_1 + \frac{1}{m} M), \quad Q^{(L)} = (I_2 - \frac{1}{m} M) \tag{3.3}$$

where the superscript R and L denote the right edge and the left edge (Fig. 1).

In the following we are going to show that the Hilbert space $\mathcal{H}_{cyl}$ can be constructed from a (non-chiral) boson theory

$$\mathcal{L} = \frac{1}{8\pi} \left[(\partial_0 \phi)^2 - (\partial_\sigma \phi)^2 \right] \tag{3.4}$$

Repeating the discussions in section 2, we may write the ϕ field as

$$\phi(t,\sigma) = \phi_0 + \tilde{\phi}_0 + p_\phi (t+\sigma) + \tilde{p}_\phi (t-\sigma) + i \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n e^{-in(t+\sigma)} + \tilde{\alpha}_n e^{-in(t-\sigma)} \right)$$

The operators ϕ_0, $\tilde{\phi}_0$, p_ϕ, $\tilde{p}_\phi$, α_n and $\tilde{\alpha}_n$ satisfy the following algebra

$$\begin{aligned} [\alpha_n, \alpha_m] =& n\delta_{n+m} \\ [\tilde{\alpha}_n, \tilde{\alpha}_m] =& n\delta_{n+m} \\ [\phi_0, p_\phi] =& [\tilde{\phi}_0, \tilde{p}_\phi] = i \\ \text{others} =& 0. \end{aligned} \tag{3.5}$$

The operator ϕ_0, p_ϕ and α_n describe the excitations on the right edge while $\tilde{\phi}_0$, $\tilde{p}_\phi$ and $\tilde{\alpha}_n$ describe the excitations on the left edge (See Fig. 1). From the last section we see that the total charge operators on the two edges are given by

$$Q^{(R)} = \frac{p_\phi}{\sqrt{l}}, \qquad Q^{(L)} = -\frac{\tilde{p}_\phi}{\sqrt{l}}. \tag{3.6}$$

(3.3) and (3.6) imply that p_ϕ and $\tilde{p}_\phi$ are quantized as

$$p_\phi = (I_1 + \frac{1}{l} M)\sqrt{l}, \qquad \tilde{p}_\phi = (-I_2 + \frac{1}{l} M)\sqrt{l} \tag{3.7}$$

The charged operators in the boson theory have a form : $e^{i\gamma\phi_L(t+\sigma)+i\tilde{\gamma}\phi_R(t-\sigma)}$:. In order for the charged operators to consist with quantization condition (3.7), γ and $\tilde{\gamma}$ must be quantized. The allowed values of γ and $\tilde{\gamma}$ are given by

$$\gamma = (n_1 + \frac{1}{l}n_3)\sqrt{l}, \qquad \tilde{\gamma} = (-n_2 + \frac{1}{l}n_3)\sqrt{l} \tag{3.8}$$

where $n_i,\ i=1,2,3$ are three integers. The operator : $e^{i\sqrt{l}\phi_L(t+\sigma)}$: (: $e^{i\sqrt{l}\phi_R(t-\sigma)}$:) create an electron on the left (right) edge, while the operator : $e^{i\phi(t,\sigma)/\sqrt{l}}$: transfer $\frac{1}{l}$ charge from one edge to the other.

The Hilbert space that satisfy the quantization condition (3.7) is generated by operators α_n, $\tilde{\alpha}_n$, $e^{i\sqrt{l}\phi_0}$, $e^{i\sqrt{l}\tilde{\phi}_0}$ and $e^{i(\phi_0+\tilde{\phi}_0)/\sqrt{l}}$. The operators α_n and $\tilde{\alpha}_n$ generate the irreducible representation of the two K-M algebras, $\mathcal{H}^{(R)}_{KM}\otimes\mathcal{H}^{(L)}_{KM}$. The total Hilbert space is given by $\mathcal{H}^{(R)}_{KM}\otimes\mathcal{H}^{(L)}_{KM}\otimes\mathcal{H}_{p\tilde{p}}$ where $\mathcal{H}_{p\tilde{p}}$ is spanned by the states $|I_1, I_2, M\rangle$ with $I_1, I_2 =$ integers and $M = 1,\ldots,l$. The charges of the state $|I_1, I_2, M\rangle$ are given in (3.3). At this stage it is not difficult to see that the Hilbert space constructed above is identical to $\mathcal{H}_{cyl} = \mathcal{H}_{disc}\otimes\mathcal{H}_{disc}\otimes\mathcal{H}_{glo}$. We conclude that the edge excitations on the two edges of the cylinder are described by the Lagrangian (3.4) and the quantization condition (3.7).

Using the effective theory (3.4), we are able to study the quantum interference effects when the two edges are brought together to form a torus. Notice that only electrons can tunnel between two edges. The operators which transfer an electron from one edge to the other is given by $e^{\pm i\sqrt{l}\phi}$. After including the electron tunneling between two edges, the system is described by the following low energy effective Lagrangian

$$\mathcal{L} = \frac{1}{8\pi}\left[(\partial_0\phi)^2 - (\partial_\sigma\phi)^2 + g\cos(\sqrt{l}\phi)\right] \tag{3.9}$$

where g measures the strength of the electron tunneling. (3.9) is the standard sine-Gordon theory (or the clock model).[12] The charged operator $e^{i\gamma\phi}$ has a dimension γ^2. Therefore, the operator $\cos(\gamma\phi)$ is relevant if $\gamma < \sqrt{2}$ and irrelevant if $\gamma > \sqrt{2}$. The operator $\cos(\sqrt{l}\phi)$ is relevant only when $l = 1$ (*i.e.,* for $\nu = 1$ quantum Hall states). In this case an arbitrary small electron tunneling will open a finite energy gap to edge excitations. But for $l \geq 3$ the operator $\cos(\sqrt{l}\phi)$ is irrelevant. It can open an energy gap only when g is greater than a finite critical value. This is a very non-trivial result. It would be interesting to observe this gap opening phase transition in the type of experiments discussed in Ref. 13.

The system (3.9) has many degenerate ground states $|m\rangle$ after the energy gap is opened. Those ground states are characterized by $\langle m|e^{i\phi/\sqrt{l}}|m\rangle = e^{i2\pi\frac{m}{l}}$ which correspond to the minimums of the potential $\cos(\sqrt{l}\phi)$. Different ground states are related by the operator $U = e^{i2\pi p_\phi/\sqrt{l}} = e^{i2\pi\tilde{p}_\phi/\sqrt{l}}$ (See (3.7)):

$$U|m\rangle = |m+1\rangle$$

From the quantization condition (3.7) we find that $|m\rangle = |m+l\rangle$. Therefore the ground states in our system are l fold degenerate. This is consistent with the well known result that the $\nu = 1/l$ FQH state has l degenerate ground states on a torus.[14,15] The solitons (kinks) in the sine-Gordon theory (3.9) carry charge $\frac{e}{l}$ and correspond to the quasi-particles in the FQH states. As a soliton propagate all the way around the circle, it will transform the ground state $|M\rangle$ into $|M+1\rangle$. This also agrees with the results in Ref. 15.

From the above example we see that the properties of the edge excitations and the properties of the FQH states on compactified space are closely related. We emphasize that this relation is very important. We know that the properties of the FQH states on compactified spaces can be used to characterize the hierarchy structures, or more precisely, the topological orders in the FQH states.[16,15] Because of the above relation, the properties of the edge excitations can also be used to characterize the topological orders in the FQH states. The dynamical properties of the edge excitations provide a practical way to experimentally measure the topological orders in the FQH states. Using this relation one should also be able to determine properties of the edge excitations from the properties of the FQH states on compactified spaces.

IV. CONCLUSIONS

In this paper we derive the effective theory of the edge excitations in the FQH states. In particular we discuss the properties of the charged excited states. The edge excitations are shown to form a new kind of states which is not described by Fermi liquid theories. Such new states are called chiral Luttinger liquids. The χLL are closely related to Fermi liquids. Actually it can be shown that the charge zero sector of the χLL is identical to the charge zero sector of a charge q Fermi liquid . Here q is the optical charge of the χLL.

The χLL is described by the chiral boson theory (2.4–2.5) and (2.22). The chiral boson theory is exactly soluble. Using this effective theory we can easily obtain all the low energy properties of the edge excitations. We calculated the ?electron propagator and the spectral function. The electron tunneling and the photoemission experiments can be used to demonstrate the non-Fermi liquid behaviors of the χLL.

Using the effective theory, we studied the interference effects between excitations on different edges. We demonstrate that the properties of the edge excitations are closely related to the properties of the FQH states on compactified spaces. The properties of the edge excitations can be used to characterize the hierarchy structures, or the topological orders in the FQH states. Using this relation we can also derive the properties of the edge excitations from the properties of the FQH states on compactified space.[17]

Another non-trivial result obtained from the chiral boson theory is that the FQH states with $\nu \neq \frac{1}{l}$ must contain more than one branches of edge excitations. This result is very general. It is independent of the edge potentials, electron interactions, *etc.* .

The dynamical properties of edge excitations contain very rich structures which reflect the rich topological orders in the FQH states. Experimental and theoretical studies of edge excitations may lead to a much deeper understanding of the FQH states and may open a new era in the FQH theory.

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FIGURE CAPTIONS

Figure 1: A FQH state on a cylinder with magnetic flux Φ.