

Chiral Spin States and Superconductivity[†]

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ABSTRACT: It is shown that several different order parameters can be used to characterize a type of P - and T -violating state for spin systems, that we call chiral spin states. There is a closely related, precise notion of chiral spin liquid states. We construct soluble models, based on P and T symmetric local spin Hamiltonians, with chiral spin ground states. Mean field theories leading to chiral spin liquids are proposed. Frustration is essential in stabilizing these states. The quantum numbers of quasiparticles around the chiral spin liquids are analyzed. They generally obey fractional statistics. Based on these ideas, it is speculated that superconducting states with unusual values of the flux quantum may exist.

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INTRODUCTION

In two spatial dimensions new possibilities arise for quantum statistics. Indeed, it has gradually emerged that fractional statistics,^{1,2,3} and the related phenomena of statistical transmutation, are rather common features of quantum field theories in two space dimensions. They can occur in σ -models with Hopf terms,² or in gauge theories with Chern-Simon terms.³ They also appear naturally in a variety of effective field theories resulting from integrating out massive 2-component fermions,^{4,5,6} for example in 2+1 dimensional quantum electrodynamics.

The general idea of fractional statistics, and the canonical construction of their description in local field theory, have been given a firm mathematical basis recently in beautiful work by Fröhlich and Marchetti.⁷ The relevant field theories can be fully regulated, and even realized on a lattice.

Given all this, one cannot help but wonder whether Nature has chosen to make use of the new theoretical possibilities in real materials. Actually, there is already compelling (though indirect) evidence that quasi-particle excitations around fractional quantum Hall effect (FQHE)⁸ states do obey fractional statistics.^{9,10} But there is an important factor which constrains the possible applications. That is, the existence of fractional statistics generically requires violation of the discrete symmetries P and T . This is because under a P or T transformation a particle with θ statistics must transform into one with $-\theta$ statistics. Generically, there will not be such a particle in the theory. Exceptions occur only if the statistical parameter is equal to zero modulo π (bosons or fermions); or if there is a doubling of the spectrum, with each particle having a partner of the opposite statistics. There is no conflict between this observation and the appearance of fractional statistics in the FQHE, because the FQHE occurs in the presence of a strong external magnetic field, which of course violates P and T .

Nevertheless, there has been much speculation recently that similar ideas apply also to the metallic oxide layers, that play a crucial role in the dynamics of high-temperature superconductors.^{11,12} Such speculation has taken various forms. One suggestion is that there is transmutation of the hole statistics, which turn these quasiparticles into bosons.¹³ Superconductivity is then pictured as a Bose condensation. Another suggestion is based on approximate mappings of spin Hamiltonians onto the Hamiltonian of the quantized Hall effect.¹⁴ Then much of the theory of the latter effect, including fractional statistics, carries over to spin systems. This idea, and closely related ideas concerning possible “flux phases”^{15,16} in the Hubbard model, will be elaborated and sharpened below. Yet another suggestion is that at certain densities a lattice of quasiholes forms, and induces diamagnetic currents whose effect mimics that of a Chern-Simons interaction.¹⁷

A common feature of several of these proposals, is that they escape the constraint mentioned above by invoking (implicitly or explicitly) spontaneous macroscopic violation of the discrete symmetries P and T , but in such a way that PT symmetry remains unbroken.^{18,19} There are prospects for direct experimental tests of this symmetry pattern.²⁰

In this note we do three things. First, we characterize the common essence of the proposed P and T violating states, which we call generically *chiral spin* states, in a precise way. We do this, by defining a local order parameter. We shall also be able, within this framework, to give one precise meaning to the notion of a spin liquid.^{21,22} Basically, a spin liquid is a chiral spin state that supports a non-local extension of the order parameter.

Second, we construct a family of spin Hamiltonians whose ground state may be found exactly, and is a chiral spin state. Our Hamiltonians, although local and simple in structure are rather contrived. Nevertheless our construction provides an existence proof for chiral spin states. In the course of constructing our model ground states we shall learn some interesting lessons about the sorts of states that support chiral spin order, and derive some intuition about when such states are likely to be energetically favorable. Unfortunately, the simple chiral spin states that diagonalize our toy Hamiltonians are not spin liquids.

So, third, we formulate some models that are not exactly soluble but do plausibly seem to have chiral spin liquid ground states. These models contain a parameter n , such that for $n=2$ they are frustrated spin models, while for large n they are tractable, in the sense that a mean field theory description is accurate. In the mean field approximation, we find chiral spin liquid states are energetically favorable for a wide range of couplings. We construct an effective field theory for the low-energy excitations around a specific chiral spin liquid state, and characterize the charge, spin, and statistics of the quasiparticles. Spin 1/2 neutral particles carrying half-fermion statistics are found, in agreement with Laughlin's arguments.²³

We conclude with some remarks on the possible relationship between chiral spin liquids and superconductivity, and put forward a speculation that flows naturally from this circle of ideas, and if true would have dramatic experimental consequences.

CHARACTERIZATION OF CHIRAL SPIN STATES

Part of the reason why the recent literature on possible dynamical realizations of fractional statistics often appears so diffuse and confusing, is that the essential character of the proposed states can be stated in several apparently different ways. Here are four possibilities, appropriate to the context of Hubbard models:

- i) As a straightforward spin ordering. Consider, in a model of spins 1/2, the expectation value

$$E_{123} \equiv \langle \sigma_1 \cdot (\sigma_2 \times \sigma_3) \rangle \quad (1)$$

where 1,2,3 label lattice sites. P symmetry would force E_{123} to vanish (or depend on the position of 123 in the lattice – see below) because it reverses the orientation of the circuit 1-2-3, which changes the sign of the triple product. T symmetry would force E_{123} to vanish. But a non-zero value of E_{123} , necessarily real because it is the expectation value of an Hermitian operator, is consistent with PT symmetry. A non-vanishing real expectation value E_{123} , correlated with the size and orientation of the triangle 123 but not its position on the lattice, is one characterization of chiral spin states.

- ii) Let us introduce electron creation operators $c_{i\sigma}^\dagger$ on site i , spin σ , and the operators

$$\chi_{ij} \equiv c_{i\sigma}^\dagger c_{j\sigma} \quad (2)$$

These χ operators have proved very convenient in the mean field theory of flux phases, and we shall use them in this way below. But first, we wish to consider a more abstract use of them, in formulating an order parameter.

Under a local gauge transformation, whereby an electron at site j acquires the phase $e^{i\theta_j}$, we have $\chi_{ij} \rightarrow e^{i(\theta_i - \theta_j)} \chi_{ij}$. In the half-filled Hubbard model at infinite U , exactly one electron occupies each site, so the states are gauge invariant.^{24,25} Therefore, according to general principles, a gauge-variant object like χ_{ij} cannot acquire a non-zero vacuum expectation value. Rather, the simplest gauge-invariant order parameters we can construct from χ are of the general type

$$\mathcal{P}\ell_{123} = \langle \chi_{12}\chi_{23}\chi_{31} \rangle \quad (3)$$

or

$$\mathcal{P}\ell_{1234} = \langle \chi_{12}\chi_{23}\chi_{34}\chi_{41} \rangle \quad (3')$$

where the χ 's circle a closed triangle or plaquette. And indeed, an expectation value of the latter type has been used to characterize the flux phase.

Are these plaquette order parameters related to the spin expectations E ? In fact simple calculations show that they are. Specifically, we have

$$\mathcal{P}\ell_{123} - \mathcal{P}\ell_{132} = -\frac{i}{2} E_{123} \quad (4)$$

and

$$\mathcal{P}\ell_{1234} - \mathcal{P}\ell_{1432} = \frac{i}{4} (-E_{123} - E_{134} - E_{124} + E_{234}) \quad (4')$$

Thus the chiral spin states are alternatively characterized by their supporting a difference between the expectation values for plaquettes traversed in opposite directions. This of course emphasizes their P -violating nature. An important formal advantage of this second definition of the chiral spin phase, is that it may be used away from half-filling, large U limit. It allows us, in other words, to step outside the framework of Heisenberg spin models.

- iii) As a Berry phase,²⁶ for transport of spins around a loop. Such phases are known to be a good way to characterize the FQHE. Specifically, consider the operator that transports the spins at 123 to sites 231. It is the permutation $P_{(123)}$. Using simple mathematical identities relating cyclic permutations to interchanges, and interchanges to spin operators, we find

$$\mathcal{B}_{123} \equiv \langle P_{(123)} \rangle = \langle P_{(23)}P_{(12)} \rangle = \frac{1}{4} \langle (1 + \sigma_2 \cdot \sigma_3)(1 + \sigma_1 \cdot \sigma_2) \rangle \quad (5)$$

and hence easily

$$\mathcal{B}_{123} - \mathcal{B}_{132} = \frac{i}{2} E_{123} \quad (6)$$

As stated, this definition of \mathcal{B}_{123} works only for spin models. In general, we may take $\vec{\sigma}_i \equiv c_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{i\beta}$, where on the left hand side $\vec{\sigma}$ is an operator but on the right hand side a numerical matrix, and use the last term of (5) as the definition of \mathcal{B}_{123} . In this generality, however, it can no longer be interpreted as the expectation of a transport operation. There are other simple relations among the various order parameters E , $\mathcal{P}\ell$, \mathcal{B} , *e.g.*,

$$Im \mathcal{P}\ell_{123} = -\frac{i}{4} E_{123} \quad (7)$$

$$Im \mathcal{B}_{123} = \frac{i}{4} E_{123} \quad (8)$$

- iv) As a state, around which the low-energy excitations are described by a field theory with a Chern-Simons term. This characterization of course is considerably more vague than the previous ones, but it is closely related to the immediately preceding phenomenon that phase is accumulated in transport around loops. We shall make precise connections in a specific model below.

In summary, we find that there are several apparently different, but in reality identical, characterizations of chiral spin states.

The chiral spin order parameter captures some, but not all, of the properties we would like to postulate of a quantum spin liquid. It has the desirable feature of leaving rotation and translation symmetry unbroken, but unfortunately it does not capture the long-range coherence we expect is necessary for an incompressible liquid. Inspired by analogy with the quantized Hall effect, however, we are led to the following preliminary definition. We say that we have a chiral spin liquid, when not only small triangles or plaquettes, but also large loops are ordered, in such a way that products around consecutive links enclosing a loop obey

$$\ln \left\langle \prod_{\gamma} \chi_{ij} \chi_{jk} \cdots \chi_{li} \right\rangle = -f(\gamma) + ibA(\gamma) \quad (9)$$

Here $f(\gamma)$ is a positive real function of the geometry of the loop γ (in our mean field models it will be proportional to the length of γ), but the crucial feature is the phase term proportional to the area $A(\gamma)$ enclosed by the loop. Identifying $\langle \prod_{\gamma} \chi_{ij} \chi_{jk} \cdots \chi_{li} \rangle$ loosely as a sort of Wilson loop, we can think of $bA(\gamma)$ as the flux enclosed by the loop γ . Although we shall not attempt to prove it in this paper, we expect that the crucial properties of the spin liquid, and specifically the statistics of its quasiparticle excitations, are determined by the coefficient b . The mean field theories we construct below support ground states with order of this type.

Before concluding this section, it seems appropriate to address two questions that might cause confusion. First: in what precise sense can we distinguish the “macroscopic” violation envisaged in chiral spin states from, say, the T violation in antiferromagnetism? After all, staggered magnetization is T odd. The crucial difference is, that the combination of T with another symmetry of the Hamiltonian, namely translation through one lattice spacing, leaves the antiferromagnetic ground state invariant. Since such a lattice translation is invisible macroscopically, the antiferromagnetic ground state is effectively T symmetric macroscopically. (In contrast, a ferromagnetic ground state does of course violate T macroscopically.)

Second: are flux phases necessarily P and T violating? Let us define this question more precisely. It is no surprise to find P and T violation in the FQHE, since there is a strong external magnetic field applied to the sample. Now we can loosely describe chiral spin states as characterized above, and the closely related flux phase states in the literature, by saying that a sort of fictitious magnetic field has developed spontaneously. Indeed, the effect of a magnetic field is precisely to modulate the phase of the wavefunction as a charged particle is transported around a loop, as in iii) above. However, we must not be too quick to infer P and T violation from this analogy. In particular, in Refs. 15,16 flux phases are constructed, in which half a fluxoid of magnetic field pierces each plaquette of a square lattice, corresponding to $\mathcal{P}\ell_{1234} = \text{negative}$. But from this alone, we cannot infer P or T violation. Indeed, the action of P or T is to change the half-fluxoid per plaquette to minus one half-fluxoid. However, $e^{i\pi} = e^{-i\pi}$ and this change is equivalent to adding a full negative fluxoid to the original configuration, which is merely a gauge transformation. Hence these symmetries should be maintained. Yet if we follow the authors of Ref. 14,27 by approximating the lattice wavefunction, in an apparently natural way, by a continuum

wavefunction, the effective flux through a loop becomes proportional to the area of the loop, and *does* generically show complex phases, indicative of P and T violation. This passage to the continuum and to an area law is necessary, if the state is to be a spin liquid in our sense (and, we suspect, in any reasonable sense.) It is not unreasonable, however, to be suspicious of an approximation that alters symmetry. The constructions which follow, were largely motivated by a desire to clarify this issue.

SOLUBLE MODEL

We have identified, and characterized in an abstract way, what we mean by a chiral spin state. We will argue below that the ground state of a frustrated Heisenberg antiferromagnet, treated in a mean field approximation, may be a chiral spin state. However, the validity of the mean field approximation in the present context is far from clear. Ideally, we would like to solve a realistic model exactly, and demonstrate that it possesses a chiral spin phase. In practice, this poses formidable problems at two levels – in formulating a realistic Hamiltonian, and in solving it. In this section, we take a different, more modest, approach.

We will presently construct a Hamiltonian, whose ground state can be explicitly identified, and is a chiral spin state (although not a spin liquid). One purpose of this exercise is to furnish an existence proof: there exists at least one Hamiltonian whose ground state violates T and P . Another is to supply us with concrete wavefunctions to look at, so that intuitions may be formed and conjectures tried.

To begin, consider four spins. These may be combined into a singlet in two different ways. Accordingly, the general wavefunction for a singlet may be written in the form:

$$(1 + \nu) |I\rangle + (1 - \nu) |II\rangle - 2|III\rangle \tag{10}$$

where

$$\begin{aligned} |I\rangle &= |\uparrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle \\ |II\rangle &= |\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle \\ |III\rangle &= |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle \end{aligned} \tag{11}$$

containing a parameter ν .

The question arises: is there any intrinsic way to separate the two-dimensional space of singlet states into one-dimensional subspaces? In fact we can think of three ways, all of which lead to the same separation:

i) We may demand, that the different spin configurations each have equal weight; that is, that the squares of their coefficients are all equal. This is the sort of situation we might expect in a liquid, where there are frequent fluctuations in the spins, but all preserving the overall spin-0 character. It is easy to see, that equality of amplitude occurs if and only if $\nu = \pm i\sqrt{3}$.

ii) We may try to impose some symmetry requirement. While one quickly realizes that our two-dimensional space is irreducible under T or under the complete group of permutations, it is easy to check that it reduces under the group of *even* permutations. The invariant subspaces, are spanned by the states with $\nu = \pm i\sqrt{3}$.

iii) We may label states by *chirality*. This is the most useful for our immediate purposes, and we now spell it out in detail.

Consider again the hermitean operator

$$\chi = \vec{\sigma}_1 \cdot (\vec{\sigma}_2 \times \vec{\sigma}_3) \quad (12)$$

At this stage, we just have a problem of three spins, each with spin 1/2. An easy computation shows that

$$\chi^2 = -4(\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2 + 15 \quad (13)$$

where $\vec{S}_i = \vec{\sigma}_i/2$, $i = 1, 2$, and 3 . Out of three 1/2 spins, we can form a spin 3/2 multiplet, which we denote by $|S = 3/2, S_z\rangle$ and two different spin 1/2 multiplets, which we denote by $|S = 1/2, S_z\rangle^+$ and $|S = 1/2, S_z\rangle^-$. We have

$$\begin{aligned} \chi^2 |S = 3/2, S_z\rangle &= 0 \\ \chi^2 |S = 1/2, S_z\rangle^\alpha &= 12, \quad \alpha = + \text{ or } - \end{aligned} \quad (14)$$

Since $|S = 1/2, S_z\rangle^\alpha$ must be orthogonal to $|S = 3/2, S_z\rangle$ we can write down

$$|S = 1/2, S_z = 1/2\rangle^+ = \frac{1}{\sqrt{3}} (|\uparrow\uparrow\downarrow\rangle + \omega|\uparrow\downarrow\uparrow\rangle + \omega^2|\downarrow\uparrow\uparrow\rangle) \quad (15)$$

and by applying the spin lowering operator

$$|S = 1/2, S_z = -1/2\rangle^+ = -\frac{1}{\sqrt{3}} (|\downarrow\downarrow\uparrow\rangle + \omega|\downarrow\uparrow\downarrow\rangle + \omega^2|\uparrow\downarrow\downarrow\rangle) \quad (16)$$

Here ω denotes the cube root of unity so that $1 + \omega + \omega^2 = 0$. Evidently, the other states $|S = 1/2, S_z\rangle^-$ are obtained from (15–16) by replacing ω by ω^2 .

The operator χ commutes with \vec{S} and thus χ, S, S_z can be simultaneously diagonalized. We find that $|S = 1/2, S_z = 1/2\rangle^+$ is an eigenstate of χ with eigenvalue $2i(\omega - \omega^2) = -2\sqrt{3}$. Evidently, $|S = 1/2, S_z = 1/2\rangle^-$ is an eigenvalue of χ with eigenvalue $2i(\omega^2 - \omega) = 2\sqrt{3}$. Note that the time reversal operator T takes $|S = 1/2, S_z = 1/2\rangle^+$ into $|S = 1/2, S_z = -1/2\rangle^-$.

Now we picture the three spins \vec{S}_1, \vec{S}_2 , and \vec{S}_3 as sitting on three of the corners of a plaquette on a square lattice. Let us couple in the fourth spin \vec{S}_4 to form a total spin singlet. From general principles, we know that two different spin singlets are possible, namely

$$|S = 0\rangle^\alpha = |S = 1/2, S_z = 1/2\rangle^\alpha \otimes |\downarrow\rangle - |S = 1/2, S_z = -1/2\rangle^\alpha \otimes |\uparrow\rangle \quad (17)$$

for $\alpha = \pm 1$.

The two $S = 0$ states are thus

$$\begin{aligned} |S = 0\rangle^- &= |\uparrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle + \omega|\uparrow\downarrow\uparrow\downarrow\rangle \\ &\quad + \omega|\downarrow\uparrow\downarrow\uparrow\rangle + \omega^2|\downarrow\uparrow\uparrow\downarrow\rangle + \omega^2|\uparrow\downarrow\downarrow\uparrow\rangle \end{aligned} \quad (18)$$

and $|S = 0\rangle^-$, obtained from $|S = 0\rangle^+$ by replacing ω by ω^2 . By construction, these states are eigenstates of $\chi = \vec{\sigma}_1 \cdot (\vec{\sigma}_2 \times \vec{\sigma}_3)$, where

$$\chi |S = 0\rangle^\alpha = 2\sqrt{3}\alpha |S = 0\rangle^\alpha, \quad \alpha = \pm 1 \quad (19)$$

Thus we have arrived again at the same separation, as promised. Since our two states are invariant under even permutations of the four spins, *i.e.*, under permutations in the classes (12) (34) and (123), and go into each other under odd permutations, *i.e.*, under permutations in the class (12) and (1234), they are also eigenstates of the other possible chirality operators, such as $\tilde{\chi} = \vec{\sigma}_1 \cdot (\vec{\sigma}_2 \times \vec{\sigma}_4)$, obtained from χ by permutations. In particular,

$$\tilde{\chi} |S = 0\rangle^\alpha = (-)2\sqrt{3}\alpha |S = 0\rangle^\alpha \quad (20)$$

Now we are ready to construct a Hamiltonian H that has a chiral spin state as ground state. The construction is best described by referring to Fig. 1. We select out of all the plaquettes on a square lattice a subset consisting of non-touching plaquettes in such a way that the corners of these plaquettes cover all the sites on the lattice. (These plaquettes are shown shaded in the figure.) We label the sites as in the figure. Now let

$$H_1 = J \left[(\vec{S}_1 + \vec{S}_2 + \vec{S}_3 + \vec{S}_4)^2 + (\vec{S}_5 + \vec{S}_6 + \vec{S}_7 + \vec{S}_8)^2 + \dots \right] \quad (21)$$

Clearly, the ground state of this Hamiltonian is reached by forming the four spins on each shaded plaquette into a singlet. Namely, the ground state is given by an infinite direct product, denoted schematically

$$\bigotimes_{\text{shaded plaquette}} (c_+ |S = 0\rangle^+ + c_- |S = 0\rangle^-) \quad (22)$$

Since on each plaquette, we can take an arbitrary linear combination of $|S = 0\rangle^\pm$, the ground state is infinitely degenerate. Let us now introduce an interaction between neighboring plaquettes by writing

$$H_2 = K [\chi(123) \chi(567) + \chi(123) \chi(9\ 10\ 11) + \dots] \quad (23)$$

The notation is the obvious one: by $\chi(567)$ we mean $\vec{\sigma}_5 \cdot (\vec{\sigma}_6 \times \vec{\sigma}_7)$, and so on.

Consider the Hamiltonian $H = H_1 + H_2$. Clearly, for small K , this describes an Ising system since on each shaded plaquette the associated ‘‘Ising spin’’ can either be up (*i.e.*, the four spins on that plaquette form $|S = 0\rangle^+$) or be down (*i.e.*, the four spins form $|S = 0\rangle^-$).

Evidently, the ground state of H for $K < 0$ is two-fold degenerate and is a chiral spin state. T and P are spontaneously broken. Notice that at high temperatures, above the usual Ising phase transition, T and P are restored.

Clearly, many other choices for the Hamiltonian are possible. For instance, in addition to H_2 , or instead of H_2 , we can add

$$H'_2 = K' [\chi(124) \chi(567) + \chi(124) \chi(9\ 10\ 11) + \dots] \quad (24)$$

For $K' < 0$, H'_2 describes an ‘‘antiferromagnetic’’ Ising interaction. In this case, the expectation value of the order parameter χ would have staggered values and macroscopically there would be no time reversal violation.

As we emphasized, our goal in this section has been to exhibit specific P and T invariant spin Hamiltonians whose ground state is a chiral spin state. The Hamiltonians we exhibited involve six spin interactions and are rather artificial. The ground state wavefunctions are rather attractive, however. In forming them we are led to add together many different spin configurations with coefficients that are equal in magnitude. This certainly calls to mind a spin liquid picture, although to induce the non-local ordering necessary for a true spin liquid would require coupling the different squares together in a less trivial way.

FRUSTRATION AND A CHIRAL SPIN LIQUID IN MEAN FIELD THEORY

Consider a two dimensional spin-1/2 Heisenberg model on a square lattice with both nearest neighbor and next-nearest neighbor along a diagonal (NNN) couplings

$$H = +J \sum_{n.n.} \vec{S}_i \cdot \vec{S}_j + J' \sum_{n.n.n.} \vec{S}_i \cdot \vec{S}_j \quad (25)$$

where J and J' parametrize the strength of the nearest neighbor and next-nearest neighbor couplings respectively. In absence of the NNN coupling computer simulations suggest that the ground state of the Heisenberg model is an anti-ferromagnetic (Néel) state which violates neither T nor P . But the NNN coupling ($J' > 0$) introduces frustrations in the Néel state. For large enough J' the Néel state is no longer favored and the ground state is expected to be a disordered state.²⁸ We will see later that such a disordered state is quite likely to be a chiral spin state. Two considerations suggest that there is a close relation between the NNN coupling and the chiral spin state. First, as discussed in the Introduction the chiral spin state may be characterized by the non-vanishing phase of the vacuum expectation value $\sigma_1 \cdot (\sigma_2 \times \sigma_3)$. This vacuum expectation value is closely related to the amplitude for moving a spin around the triangle 123. It is plausible that the next-nearest neighbor coupling frustration induces coherent hopping of spins around the triangle and allows the operator $\sigma_1 \cdot (\sigma_2 \times \sigma_3)$ to develop a nonzero vacuum expectation value. Second, in Ref. 27 it has been suggested that the flux phase on square lattice may be regarded as an alternative realization of the spin liquid state constructed by Kalmeyer and Laughlin¹⁴ on a triangular lattice. However, the flux phase respects T and P while the Kalmeyer-Laughlin state does not. This is because in the flux phase $\mathcal{P}\ell_{1234} = e^{i\pi}$ and under T and P $\mathcal{P}\ell_{1234} \rightarrow \mathcal{P}\ell_{1234}^* = \mathcal{P}\ell_{1234}$. To find a state corresponding more closely to the Kalmeyer-Laughlin state, and in particular sharing its symmetry properties, we must find a state similar to the flux phase, that in addition violates T and P . One way this can happen, is for $\mathcal{P}\ell_{123}$ in Equation (3) to develop a non-zero vacuum expectation value. The phase of $\mathcal{P}\ell_{123}$ is the flux through the triangle 123, and is equal to half the flux through the plaquette 1234, that is $e^{\frac{i\pi}{2}}$, if the ground state is homogeneous. Therefore a non-zero $\mathcal{P}\ell_{123}$ can break T and P symmetry. It is possible to construct models, which in mean field theory develop the desired vacuum expectation values. The corresponding vacuum state, as we will see, does seem to resemble the Kalmeyer-Laughlin state qualitatively.

We will now show, in a mean field approach,²⁹ that an extrapolated form of the Hamiltonian (25) supports a locally stable vacuum which breaks T and P , if J'/J is larger than a critical value. Following Refs. 16,25 we first introduce the electron destruction operator

$c_{i\alpha}$ and write the Lagrangian of the model (25) as

$$L = \sum_i c_i^\dagger (i\partial_t) c_i - H_e - \sum_i a_0(i)(n_i - 1) \quad (26)$$

where $n_i = c_i^\dagger c_i$ is the number of electrons on the i th site and the Lagrange multiplier term $\sum_i a_0(n_i - 1)$ is introduced to enforce the constraint $n_i = 1$. H_e in (26) is obtained by replacing \vec{S}_i by

$$\vec{S}_i = c_i^\dagger \vec{\sigma} c_i \quad (27)$$

and reads

$$H_e = \sum_{n.n.} 2J c_i^\dagger c_{i\beta} c_j^\dagger c_{j\alpha} + \sum_{n.n.n.} 2J' c_i^\dagger c_{i\beta} c_j^\dagger c_{j\alpha} - 2N(J + J') \quad (28)$$

The constant term $-2N(J + J')$ is included for later convenience. In the path integral formalism the partition function is given by

$$Z = \int D a_0 D c_i D c_i^\dagger e^{i \int L dt} = \int D a_0 e^{i \int L_{\text{eff}}(a_0) dt} \quad (29)$$

where $L_{\text{eff}}(a_0)$ is the effective Lagrangian obtained by integrating out electrons. $L_{\text{eff}}(a_0)$ is very complicated and we have to do a saddle point approximation here. Putting aside the question of the accuracy of this approximation for a moment, we find that the important configurations are given by the stationary points of the effective action

$$\frac{\delta L_{\text{eff}}(a_0)}{\delta a_0} = 0 \quad (30)$$

The energy of the approximate ground state, constructed in this way, is given by $-L_{\text{eff}}(a_0)$ at the stationary point. From (26) it is not difficult to see that the mean field energy $-L_{\text{eff}}(a_0)$ is equal to the vacuum energy of the Hamiltonian

$$H_1 = H_e + \sum_i a_0(n_i - 1) \quad (31)$$

where the electron operators are now no longer subject to the constraint $c_i^\dagger c_i = 1$. H_1 is still very difficult to solve, but since the constraint $n_i = 1$ is removed we can easily use the variational method (i.e., effectively the Hartree-Fock approximation) to find a state close to the true vacuum of H_1 .

Here we are primarily interested in the spatially homogeneous stationary point $a_0 = \text{const}$. The constant a_0 acts like a chemical potential in (31). From (30) and (31) it is not

hard to see that the stationary value of a_0 is such that the total number of electrons in ground state of H_1 is equal to the number of the lattice sites N . This of course corresponds to our starting point, which was a model with one spin degree of freedom per site.

As our trial wave function for H_1 , let us first consider the bond state studied by Affleck and Marston.¹⁶ The bond state is defined as the ground state of the following quadratic Hamiltonian:

$$H_B = \sum_{n.n.} \left(\tilde{\chi}_{ij} c_j^\dagger c_i + h.c. \right) + \sum_{n.n.n.} \left(\tilde{\chi}_{ij} c_j^\dagger c_i + h.c. \right) \quad (32)$$

with a total number of N electrons. We must vary the $\tilde{\chi}_{ij}$'s to minimize the energy the ground state.

Following Affleck and Marston we will consider the $\tilde{\chi}_{ij}$'s which break the symmetry under translation by one lattice spacing, but are invariant under translations by two lattice spacings. These $\tilde{\chi}_{ij}$'s are parametrized by eight complex parameters. The nearest neighbor hopping amplitudes $\tilde{\chi}_{ij}$ are parametrized by $\tilde{\chi}_i$, $i = 1, \dots, 4$ in the way described in Ref. 16. The next-nearest neighbor hopping amplitudes are given by

$$\begin{aligned} \tilde{\chi}_{i,i+\hat{x}+\hat{y}} &= \tilde{\chi}_5 + (-)^i \tilde{\chi}_6 \\ \tilde{\chi}_{i,i+\hat{x}-\hat{y}} &= \tilde{\chi}_7 + (-)^i \tilde{\chi}_8 \end{aligned} \quad (33)$$

In momentum space H_B can be written as

$$H_B = \sum'_k \psi_k^\dagger h_k \psi_k \quad (34)$$

where \sum'_k is a summation over half of the Brillouin zone and $\psi_k = \begin{pmatrix} c_k \\ c_{k+Q} \end{pmatrix}$. h_k is given by

$$h_k = 2 \operatorname{Re} \eta_1 + 2 \operatorname{Re} \eta_2 \tau_1 + \operatorname{Re} \eta_3 \tau_3 + \operatorname{Im} \eta_3 \tau_2 \quad (35)$$

where

$$\begin{aligned} \eta_1 &= \tilde{\chi}_5 e^{i(k_x+k_y)} + \tilde{\chi}_7 e^{i(k_x-k_y)} \\ \eta_2 &= \tilde{\chi}_6 e^{i(k_x+k_y)} + \tilde{\chi}_8 e^{i(k_x-k_y)} \\ \eta_3 &= \tilde{\chi}_1 e^{ik_x} + \tilde{\chi}_2^* e^{-ik_y} + \tilde{\chi}_3 e^{-ik_x} + \tilde{\chi}_4^* e^{ik_y} \end{aligned} \quad (36)$$

In (36) we have taken the lattice constant $a = 1$.

Now we can diagonalize H_B . The energy spectrum is given by

$$E_k = 2\operatorname{Re}\eta_1 \pm \sqrt{4(\operatorname{Re}\eta_2)^2 + |\eta_3|^2} \quad (37)$$

In the absence of the diagonal hopping terms ($\tilde{\chi}_i|_{i=5,\dots,8} = 0$) the Fermi ‘‘surface’’ of E_k (at half filling) consists of just the two isolated points at $k = (\pi/2, \pi/2)$ and $k = (\pi/2, -\pi/2)$. The low lying excitations around this Fermi ‘‘surface’’ correspond to two families of massless Dirac fermions in the continuum limit. Each family contains a spin up and a spin down electron. However, the non-zero diagonal hopping terms with $\tilde{\chi}_6 = -\tilde{\chi}_8 = \text{real}$ open a gap

at the Fermi “surface”. Since generically it is energetically favorable for fermion systems to open a gap at the Fermi surface, we expect $\tilde{\chi}_6$ and $\tilde{\chi}_8$ to develop a non-zero value if J' is large enough. In the following we will show, in a mean field approximation, this is exactly what happens.

The ground state $|\Phi_B\rangle$ of H_B can be obtained by filling all negative energy levels with electrons. Using $|\Phi_B\rangle$ we can obtain $\langle c_i^\dagger c_j \rangle = \chi_{ij}$ for nearest neighbor and next-nearest neighbor bonds. Those χ_{ij} 's are again parametrized by eight complex parameters χ_i , $i = 1, \dots, 8$ corresponding one by one to the $\tilde{\chi}$'s. In fact the χ_i 's can be expressed as derivatives of the ground state energy of H_B with respect to the $\tilde{\chi}_i$'s. In terms of χ_i 's the expectation value of H_1 (see (31)) in the state $|\Phi_B\rangle$ can be written as

$$E_B = \langle \Phi_B | H_1 | \Phi_B \rangle = -N \left[J \left(|\chi_1|^2 + |\chi_2|^2 + |\chi_3|^2 + |\chi_4|^2 \right) + J' \left(|\chi_5 + \chi_6|^2 + |\chi_5 - \chi_6|^2 + |\chi_7 + \chi_8|^2 + |\chi_7 - \chi_8|^2 \right) \right] \quad (38)$$

Since χ_i 's are functions of $\tilde{\chi}_i$'s we can adjust $\tilde{\chi}_i$'s to minimize the energy E_B . By computer search we find that there are two local minima which are potential ground states. One is a chiral spin state, characterized by

$$\begin{aligned} \chi_i |_{i=1,\dots,4} &= f e^{i\frac{\pi}{4}} \\ \chi_5 &= \chi_7 = 0 \\ \chi_6 &= -\chi_8 = \pm g \neq 0 \end{aligned} \quad (39)$$

where f and g are real constants. One can easily check the flux through the triangles is $+\frac{\pi}{2}$ ($-\frac{\pi}{2}$) for $\chi_6 = -\chi_8 = +g$ ($-g$). In this state, P and T are broken. The chiral spin state exists only for $J'/J \gtrsim 0.5$. When $J'/J \lesssim 0.5$ we find $g = 0$ and the chiral spin state is bettered by the flux phase discussed in Refs. 15,16. P and T are not broken in this flux phase.

Still within the framework of bond states, another local minimum is the dimer phase characterized by

$$\begin{aligned} \chi_1 &= 1 \\ \chi_i |_{i \neq 1} &= 0 \end{aligned} \quad (40)$$

The energy (per site) of the dimer phase is equal to $-J$. The energies of the chiral spin state, the flux phase, and the dimer state are plotted in Fig. 2.

In addition to the bond states, another obvious mean field state to consider is the Néel state, characterized by

$$\begin{aligned} \langle c_i^\dagger \sigma^3 c_i \rangle &= (-)^i \\ \chi_{ij} &= 0 \end{aligned} \quad (41)$$

The energy of the Néel state is given by $-2(J - J')$. A second spin ordered state, characterized by

$$\begin{aligned} \langle c_i^\dagger \sigma^3 c_i \rangle &= (-)^{i_x} \\ \chi_{ij} &= 0 \end{aligned} \quad (42)$$

also has low energy when J' is large. We will call this state the stripe state. The mean field energy of the stripe state is $-2J'$

Thus in the mean field approximation either the Néel state (for $0 < J'/J < 0.5$) or the stripe state (for $0.5 < J'/J < 1$) always has the lowest energy. But near $J'/J = 0.5$ the chiral spin state comes very close. Actually at $J'/J = 0.5$ $E_{chiral} = -0.918$ and $E_{Neel} = E_{stripe} = E_{dimer} = -1$. Because of the expected large quantum fluctuations the saddle point or mean field approximation is inadequate to determine which, if any, of the mean field trial states actually describes the true ground state. However the above calculation at least indicates that the chiral spin state is a serious candidate for the true ground state.

As we have mentioned repeatedly, the mean field approach to the spin 1/2 Heisenberg model is not reliable. Specifically, if we take any of our trial states and go back and compute corrections to the corresponding saddle point, we shall find them to be huge. In order to have a context in which we can use the relatively tractable mean field method, and yet have a controlled approximation, we can go to an appropriate large n limit, so that there are many individuals contributing to the mean field. As is well known in many other contexts, in this limit the case the mean field approximation is at least self-consistent, order by order in $1/n$. A large n limit appropriate to our problem can be achieved by considering the Hamiltonian

$$\begin{aligned}
H_n = & 2 \sum_{n.n.} J c_i^{\dagger a} c_{ib} c_j^{\dagger b} c_{ja} \\
& + 2 \sum_{n.n.n.} J' c_i^{\dagger a} c_{ib} c_j^{\dagger b} c_{ja} \\
& - n(J + J')N
\end{aligned} \tag{43}$$

where $a, b = 1, \dots, n$. In the ground state, H_n contains $nN/2$ fermions. For $n = 2$, H_n reduces to H_1 in (31).

We may repeat our previous calculations for the energies of bond states. The energies of the chiral spin state and the dimer state are the same as before except for an overall factor $n^2/4$.

Now let us consider the staggered phase (corresponding to the Néel phase for $n = 2$), characterized by

$$\begin{aligned}
\langle c_i^{\dagger a} c_{ib} \rangle &= \frac{1}{2} \delta_b^a + (-)^i T_b^a \\
\chi_{ij} &= 0
\end{aligned} \tag{44}$$

where T_b^a is a traceless hermitian matrix. The minimum of the energy is at

$$\begin{aligned}
 T &= \begin{pmatrix} \frac{1}{2} & & & & & \\ & \ddots & & & & \\ & & \frac{1}{2} & & & \\ & & & -\frac{1}{2} & & \\ & & & & \ddots & \\ & & & & & -\frac{1}{2} \end{pmatrix} & \text{for } n = \text{even} \\
 T &= \begin{pmatrix} \frac{1}{2} & & & & & \\ & \ddots & & & & \\ & & \frac{1}{2} & & & \\ & & & 0 & & \\ & & & & -\frac{1}{2} & \\ & & & & & \ddots \\ & & & & & & -\frac{1}{2} \end{pmatrix} & \text{for } n = \text{odd}
 \end{aligned} \tag{45}$$

The minimal energy is given by

$$\langle H \rangle_{sta} = -2N(J - J') \left[\frac{n}{2} \right] \tag{46}$$

where $[x]$ is the integer part of x . For the stripe state (and other “spin” ordered states) the mean field energy is also of order $-O(n)$. Thus in the large n limit the bond states are always favored, because their energies are of order $-O(n^2)$. There are then only two possible phases, the chiral spin state and the dimer state, in the range $0 < J'/J < 1$. The energy of the chiral spin state is slightly higher than that of the dimer state. At $J'/J = 1$ we get $E_{chiral} = -0.994J$ and $E_{dimer} = -J$ — the two energies are extremely close. Thus the chiral spin state is very likely to be a locally stable state. It is hard to imagine a path connecting the two states without encountering a potential barrier. There is an argument, relying on a result we will show later, that strongly suggests that such paths do not exist. That is, the effective action for low energy excitation around the chiral spin state contains a Chern-Simons term with integer coefficient while the effective action for excitations around the dimer phase contains no Chern-Simons term. The integer coefficient of the Chern-Simons term can jump to another value only when the gap in the electron spectrum is closed. Therefore, for any path connecting the chiral spin state and the dimer state, there must be a state along the path for which the energy gap in the electron spectrum closes. However, such a gapless state very likely has higher energy than the chiral spin state (with its gap). The gapless states, we conjecture, constitute a potential barrier between the chiral spin state and the dimer state.

The chiral spin phase will definitely become the energetically favored possibility once we consider Hamiltonians containing an additional term of the form

$$-\frac{\tau}{n^4} \sum_{ij} G_i G_j. \tag{47}$$

In (32) G_i is the order parameter discussed before (see (1) and (4)) and is given by

$$G_i = i[(c_i^\dagger c_{i+\hat{x}})(c_{i+\hat{x}}^\dagger c_{i+\hat{x}+\hat{y}})(c_{i+\hat{x}+\hat{y}}^\dagger c_i) - (i + \hat{x} \leftrightarrow i + \hat{x} + \hat{y})] \tag{48}$$

Such a term does not change the energy of the dimer phase because $G_i = 0$ in the dimer phase. But the added term lowers the energy of the chiral spin state if $\tau > 0$. The mean field phase diagram is plotted in Fig. 3.

In this section we have argued that the chiral spin state is very likely a locally stable state of the frustrated Heisenberg model, in the large n mean field approximation. For slightly modified Hamiltonians, it is plausibly the ground state. It should therefore be sensible to study the quasi-particle excitations around this state. We also found that the mean field energy of the chiral spin state is very close to that of other ordered states, *i.e.*, the Néel, stripe, and dimer states, even for the original case of $n = 2$. This suggests that the quantum fluctuation may well melt the ordered phases, resulting in a chiral spin ground state. Even if this does not happen, the chiral spin state may appear as ground state of a modified Heisenberg model. Hopping terms around the plaquette (see (47)) favor the chiral spin state. Furthermore, although we will not review it here, hopping terms of the simpler sort

$$H_{hop} = -t \sum_{\sigma, n.n.} c_{i\sigma}^\dagger c_{j\sigma} \quad (49)$$

also favor the chiral spin state. Altogether, we are encouraged to take seriously the possibility that order of this kind develops under rather general circumstances, in frustrated spin models.

The chiral spin states defined here are spin liquids, according to our definition. Indeed, in the large n mean field theory the expectation value of products of χ 's around arbitrarily large closed paths is merely the product of their nominal values on single links. Since the number of elementary triangular plaquettes enclosed by a closed path is proportional to the area enclosed, and each contributes the same constant to the imaginary part, the area law (9) is manifestly satisfied.

Finally, let us remark that although we have made life easy for ourselves by going to mean field theory, we have probably made it hard for the spin liquid. After all, we expect the liquid to be stabilized, relative to say the dimer, precisely by fluctuations, and mean field theory systematically minimizes fluctuations.

QUANTUM NUMBERS OF QUASI-PARTICLES

We now turn to discuss the quantum numbers of the quasi-particles, first qualitatively and then more formally.

As we have seen the chiral spin phase is stabilized, relative to the dimer, by hopping. Very roughly, we may say that the dimer melts as the electrons become even slightly free to wander. Presumably, this effect of including plaquette terms or of increasing t by hand also would be induced dynamically as a by-product of doping. Indeed, as one moves away from half-filling, vacant sites become available, so the electrons begin to move. In any case, it seems sensible to think of the chiral spin phase as a quantum liquid. We expect it to be incompressible, due to strong Coulomb repulsion.

In this context, the postulated area law (9) acquires a simple physical interpretation. It means that the effective magnetic flux, introduced above as a Berry phase associated above with transport around fixed loops in physical space, can instead be ascribed – much more reasonably – to the transport of particles around one another. (This ascription is

possible, if (and only if) the spin fluid is incompressible. For only then, are particle number within a loop, and the area of the loop, interchangeable.) In other words, fictitious fluxes and charges are to be attached to each particle, in such a way that the Berry phase is realized as the phase accumulated according to the Aharonov-Bohm effect for transport of these fictitious charges and fluxes around one another.

Sophisticated readers will recognize here the appearance of statistical transmutation. Indeed, the analysis here is entirely parallel to a similar one for the FQHE. Let us determine, following a slightly different path from the one laid down in that analysis, the relevant numbers. A defect in our featureless singlet spin liquid can be introduced by constraining the spin on one site to be, say, up. The density of the liquid is then reduced, because the site in question can only be reached by neighboring electrons already spinning up. In effect, one-half a site - and therefore, by incompressibility, one-half an electron - has been removed. Since the phase was $e^{i\pi}$ per encircled electron, it becomes $e^{i\pi/2}$ for encircling the defect. Now we can expect that upon our slowly delocalizing the constraint the system will relax to an energy eigenstate with spin 1/2. As long as there is a gap neither the total spin nor the phase accompanying transport around a loop far from the defect can be altered by the relaxation, which is a local process. Thus we expect that the defect relaxes into a spin 1/2, neutral, half-fermion quasi-particle.

The conclusion of the preceding highly heuristic argument can be illustrated concretely in the continuum limit of our chiral spin phase. In the flux phase the Fermi “surface” consists of two isolated points. The low energy excitations correspond to two families of fermions in continuum limit, whose propagation is described by the effective Lagrangian

$$\sum_{\substack{a=1,2 \\ \alpha=\pm}} \bar{\psi}_{a\alpha} \gamma^\mu (i\partial_\mu + a_\mu + A_\mu) \psi_{a\alpha} \quad (50)$$

where $\alpha = \pm$ labels spin up and spin down, A_μ is the electromagnetic gauge potential and γ^μ is given by

$$\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^2. \quad (51)$$

a_μ is the dynamically generated gauge potential discussed in Refs. 24 ,25 a_0 comes from the Lagrange multiplier term used to enforce the constraint $n_i = 1$. $a_i|_{i=1,2}$ comes from the phase of the hopping amplitude χ_{mn}

$$e^{-i \int_m^n \vec{A} \cdot d\vec{x}} \chi_{mn} = e^{i \int_m^n \vec{a} \cdot d\vec{x}} \quad (52)$$

The electromagnetic gauge potential is included to make the right hand side of (52) an electromagnetic gauge invariant object. In the chiral spin phase the electron spectrum opens a gap at the fermi surface. This corresponds to the fermion fields $\psi_{a\alpha}$ obtaining a mass term. We find that the mass terms obtained by $\psi_{1\alpha}$ and $\psi_{2\alpha}$ have the same sign

$$m\bar{\psi}_{1\alpha}\psi_{1\alpha} + m\bar{\psi}_{2\alpha}\psi_{2\alpha}. \quad (53)$$

Such a mass term breaks T and P , which just reflects the symmetry properties of the chiral spin phase. Putting (50) and (53) together we obtain the Lagrangian of the continuum limit of the chiral spin state

$$\mathcal{L} = \sum_{\substack{a=1,2 \\ \alpha=\pm}} \bar{\psi}_{a\alpha} \gamma^\mu (i\partial_\mu + a_\mu + A_\mu) \psi_{a\alpha} + m\bar{\psi}_{a\alpha}\psi_{a\alpha} \quad (54)$$

At half filling we can safely integrate out the massive fermions and obtain the following effective Lagrangian

$$\mathcal{L}_{\text{eff}} = 4 \frac{m}{|m|} \frac{1}{8\pi} (a_\mu + A_\mu) \partial_\nu (a_\lambda + A_\lambda) \epsilon^{\mu\nu\lambda} \quad (55)$$

The factor 4 is because of the four fermions.

Using the effective Lagrangian we may obtain the low energy properties of the chiral spin phase. First we would like to show that there is no zero magnetic field Hall effect in the chiral spin phase (*i.e.*, the conductance $\sigma_{xy} = 0$), even though it is no longer forbidden by the (broken) symmetries P and T . The electrical current is defined by

$$J_e^\mu = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial A_\mu} = \frac{1}{\pi} \frac{m}{|m|} \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda + A_\lambda) \quad (56)$$

The equation of motion for a_μ reads

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}_{\text{eff}}}{\partial a_\mu} = \frac{1}{\pi} \frac{m}{|m|} \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda + A_\lambda) \\ &= J_e^\mu \end{aligned} \quad (57)$$

This implies that the electrical current vanishes for any background electromagnetic field and the chiral spin phase at half filling is an insulator. This is hardly shocking, since our effective theory (54) or (55) is supposed to describe the low energy properties of the Heisenberg model (25), which contains no charge fluctuations. However it was not completely obvious *a priori* in our mean field approximation, which does allow charge fluctuations.

Now let us consider the excitations in the chiral spin phase. The simplest excitation to consider is an excited electron in the conduction band. We must emphasize that the appearance of an electron in the conduction band does not correspond to introducing an electron into our system, because integrating out a_0 still enforces the constraint $n_i = 1$. We will see that such an excited electron corresponds to a neutral spin 1/2 particle. At low energy the excited electron can be regarded as a test particle. The effective Lagrangian in presence of such a particle can be written as

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \frac{1}{2\pi} \frac{m}{|m|} \epsilon^{\mu\nu\lambda} (a_\mu + A_\mu) \partial_\nu (a_\lambda + A_\lambda) \\ &\quad + (a_\mu + A_\mu) j^\mu \end{aligned} \quad (58)$$

where j^μ is the current of the test particle. Now the electrical current and the equation of motion become

$$J_e^\mu = \frac{1}{\pi} \frac{m}{|m|} \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda + A_\lambda) + j^\mu \quad (59)$$

and

$$0 = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial a_\mu} = j^\mu + \frac{1}{\pi} \frac{m}{|m|} \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda + A_\lambda) \quad (60)$$

The first term in (59) can be regarded as the contribution to electrical charge arising from the vacuum polarization. The equation of motion implies that the electrical charge of the

excited electron is completely screened by vacuum polarization. The screened electron behaves like a neutral particle. Because the chiral spin vacuum is a spin singlet even when a_μ and A_μ are non zero the vacuum polarization can not change the spin quantum number of the excited electron. Therefore the screened electron is really a spin 1/2 neutral particle. Due to the Chern-Simons term in (58) the statistics of the excited electron is also changed. From Ref. 5 and 30 we find that the statistics are given by a phase factor $e^{i(\pi + \frac{\pi}{2} \frac{m}{|m|})}$. Thus the screened electron behaves like a half fermion. The quantum numbers and the statistics of the screened electron are exactly the same as the spinon in the Kalmeyer-Laughlin state, and of course the same as we obtained heuristically before.

It may well seem that there is no connection whatsoever, or even a mismatch, between our heuristic argument and our formal argument. According to the former, the quantum statistics of the quasi-particle is determined by the ratio of fictitious flux density to particle density. According to the latter, it is determined by the number of points at which the energy gap closes, if the flux is turned off. Most remarkably, however, these two quantities are related by an index theorem.³¹ We shall illustrate how this works, by considering a generalized flux phase.

The construction of the chiral spin state given above was based on the particular flux phase such that the flux through each plaquette is equal to π . A similar construction of the chiral spin state can be also done for generalized flux phase where the flux through each plaquette is equal to $2\pi p/q$, with q an even integer. It has been shown that the Fermi “surface” of such a flux phase (at half filling) consists of q isolated points, and that each point corresponds to a two-component massless Dirac fermion in the continuum limit. Including proper non-nearest neighbor hopping terms, we give each of the q pairs of fermions, connected by a perturbation at the appropriate wave vector, a mass. The mass for each family can be shown to have the same sign. Thus the generalized chiral spin state is described, in the continuum limit, by

$$\mathcal{L} = \sum_{\substack{a=1,\dots,q \\ \alpha=\pm}} \bar{\psi}_{a\alpha} \gamma^\mu (i\partial_\mu + a_\mu + A_\mu) \psi_{a\alpha} + m \bar{\psi}_{a\alpha} \psi_{a\alpha} \quad (61)$$

After integrating out the fermions we obtain a Chern-Simon term as in (58) but with the factor 4 replaced by $2q$. The neutral spin 1/2 excitations then acquire fractional statistics given by $e^{i(\pi + \frac{m\pi}{|m|q})}$. This is exactly the same result, as would follow from our heuristic argument.

CONCLUDING REMARKS

Now let us briefly discuss what all this might have to do with high-temperature superconductivity. A spinon of the type described above, carrying half-fermion statistics, plausibly binds to any introduced hole, creating a spinless charged half-fermion composite. Two half-fermions can pair to make a boson, and such boson pairs are good candidates for a superconducting condensate. The pairing is energetically desirable, because a *pair* of introduced holes, generate a fictitious flux which is an integral multiple of the fundamental fluxoid. In other words a pair can peacefully coexist with the chiral spin phase background, and therefore need not carry spinons along.³² The qualitative idea here is not altogether unlike that underlying “spin bag”³³ or “spin polaron” mechanisms. According

to these pictures too, holes are associated with disordered patches, and so it is advantageous to minimize their effect by clumping them together. There is a significant difference, however: the present mechanism does not require an antiferromagnetic Néel or spin wave background to play against.

Another related argument for superconductivity in doped chiral spin liquids, given by Laughlin,^{12,34} is the following. It is known that fermions with arbitrarily weak attraction become superconducting at zero temperature. Now half-fermions can be considered as fermions with a special sort of long-range attraction. Thus, they must condense at low temperature.

We conclude with some philosophy and a speculation. The message of this paper, and of several others in the recent literature might be phrased roughly as follows. The success of the Laughlin wavefunctions in describing the incompressible quantum liquid phases of the FQHE, shows that they provide an excellent way to reconcile the desire to order (in that context, order in real space is desired, to minimize Coulomb repulsion) with the difficulties introduced by frustration (in that case, by an external magnetic field). Roughly speaking, in the FQHE the electron gas, by collective correlations, manufactures an effective magnetic field to cancel the real one. Now a frustrated spin system faces similar problems. Let us imagine attempting to find the ground state in the usual way, by evolving the system in imaginary time. We can think of a spin sampling various loops as it decides how to point, and in general getting conflicting instructions. By condensing into a chiral spin liquid, the spins introduce collective phases, that partially ameliorate the frustration. It is no accident, then, that the sorts of effective field theories and order parameters we find for chiral spin liquids, are so reminiscent of those familiar in the quantized Hall effect. Concretely, spinon excitations around the chiral spin state near half filling, analyzed above, have the same statistics as one finds for the quasiparticles around the $m=2$ Laughlin state. We believe this conclusion, originally derived by Laughlin from an approximate mapping of the Heisenberg antiferromagnet on a triangular lattice into a quantum Hall system, is much more robust; it follows generally for chiral spin liquid states having $b = \pi n_e$ in the area law (9), where n_e is the density of electrons.

This circle of ideas strongly suggests a conjecture, that if true leads to a dramatic consequence. It is quite conceivable that in different parameter regimes, or in the real materials at different doping levels, other possibilities than $m = 2$ occur. Indeed, we have briefly discussed such possibilities before, in illustrating the consistency of our qualitative and quantitative arguments for fractional statistics of spinons. In mean field theory, these different possibilities lead to gaps opening at different places. It should be favorable to open a gap at the fermi surface, so the system might switch from one phase to another as the position of this surface changes. And, of course, a whole menagerie of states has been observed in the FQHE. Now if, for instance, an $m = 4$ chiral spin state were formed, holes doped into it might be expected to condense in quadruples, thus producing a fluxoid unit $h/4e$. (This time, we are speaking of genuine magnetic flux!). All this suggests that it would be worthwhile checking the unit of flux quantization carefully in the new materials under various circumstances, not prejudging the universality of pairing. There may be surprises lurking at different doping levels, pressures – or even simply at lower temperatures.

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FIGURE CAPTIONS

- Figure 1: The shaded plaquettes are selected to construct the Hamiltonian in Eq. (21).
- Figure 2: The mean field energies of the chiral spin state, the flux state and the dimer state.
- Figure 3: The mean field phase diagram of the Hamiltonian (43) with an added term (47).

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