Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution

Xiao-Gang Wen

Condensed matter physics has been around for 100 years
Where are we in condensed matter physics?
Where are we in physics (has been around for 400 years)?
Is condensed matter physics or physics largely finished?
Have we seen the beginning of the end of physics?
How to gain a deeper understanding of our rich world?

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Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
Discovery $\rightarrow$ Unification $\rightarrow$
More discovery $\rightarrow$ More unification $\rightarrow$ ...

• Each unification = a revolution in physics

• Each revolution introduces new mathematical language into physics

A deeper understanding of natural phenomena always requires a new mathematical language
Newton (1687)

- unified gravity (the falling cow on earth) and astronomy (the planets motions in space)
- obtained laws of mechanical motion and a quantitative theory of gravity.
- **New math: Calculus**
Maxwell (1861)
- unified electricity, magnetism, and light
- obtained a quantitative and dynamical theory of electromagnetism: the Maxwell equation
\[ \dot{E} - c \partial \times B = \dot{B} + c \partial \times E = \partial \cdot B = \partial \cdot E = 0. \]
- **New math: Fiber bundle (gauge theory)**
Relativity revolution

Einstein (1905, 1916)

- Unified space and time: the special relativity
- Unified the pull by gravity and the “pull” felt in accelerating frame
- Obtained a dynamical theory of gravity: the Einstein equation

New math: Riemannian geometry (curved space)

Michelson-Morley (1887)

Long-range entangled quantum matter and a unification of forces
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  \[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = -\frac{8\pi}{c^4} T_{\mu\nu} \]
- New math: Riemannian geometry (curved space)

Michelson-Morley (1887)
Quantum revolution

- Unified: Hydrogen spectra, blackbody radiation, double-slit interference, ...

- Unified: frequency ↔ energy, wave-length ↔ momentum

- *particle-wave reality*

- **New math:** linear algebra

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Visible spectrum

- Hydrogen
- Neon
- Iron

[Graph showing intensity vs. wavelength for different temperatures and the classical theory for 3000 K.]
What quantum theory really unifies is **matter** and **information**. Changing information (qubits) → energy → mass → matter (according to quantum physics and relativistic physics)
- Frequency → a property of information
- Energy/mass → a property of matter

Unification: The above seemingly unrelated phenomena all come from long-range entangled qubits (ie topological order, Wen 89)

New math: tensor category, group cohomology, ... → A continuation of quantum revolution, a second quantum revolution
• What quantum theory really unifies is **matter** and **information**.
  Changing information (qubits) $\rightarrow$ energy $\rightarrow$ mass $\rightarrow$ matter
  (according to quantum physics and relativistic physics)
  - Frequency $\rightarrow$ a property of information
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• But can simple qubits (quantum information) really explain/unify so many fascinating and mysterious properties of universe?
  (1) all interactions are gauge interaction
    (electromagnetism, strong/weak int. Gauge bosons in CMP?)
  (2) most matter are formed by particles with Fermi statistics
    (electrons, quarks, etc. Fractional/non-Abelian statistics in CMP)
  (3) angular momentum of fermions are fractionalized
    (spin-1/2. Fractionalization of other quantum numbers in CMP)
Discovery $\rightarrow$ Unification $\rightarrow$ More discovery $\rightarrow$ More unifi.

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- **Unification**: The above seemingly unrelated phenomena all come from **long-range entangled qubits** (ie **topological order** Wen 89)

- **New math**: tensor category, group cohomology, ... ... →

  A continuation of quantum revolution, **a second quantum revolution**
Why do we want to study long-range entanglement (= topological order)?

Because long-range-entanglement/topological-order
• give rise to new states of quantum matter, new quantum phase transitions,
• give rise to perfect conducting edge channels even with impurities,
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- explain the origin of spin-1/2, and other fractional quantum numbers,
- explain the origin of Fermi statistics and fractional statistics,
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- give rise to perfect conducting edge channels even with impurities,
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- explain the origin of spin-1/2, and other fractional quantum numbers,
- explain the origin of Fermi statistics and fractional statistics,
- unify almost all the know theories.

Let us first review our general understanding of states of matter
In primary school, we learned ...

there are four states of matter:

Solid

Liquid

Gas

Plasma
In university, we learned there are much more phases

- Rich forms of matter ⇐ rich types of orders
- A deep insight from Landau: different orders come from different symmetry breaking.
  → Symmetry breaking theory of orders
  → A corner stone of condensed matter physics
A local many-body quantum system

- A many-body quantum system
  \( = \) Hilbert space \( \mathcal{V}_{tot} + \) Hamiltonian \( H \)
- The locality of the Hilbert space:
  \( \mathcal{V}_{tot} = \bigotimes_{i=1}^{N} \mathcal{V}_i \)
- The \( i \) also label the vertices of a graph
- A local Hamiltonian \( H = \sum_x H_x \) and \( H_x \) are local hermitian operators acting on a few neighboring \( \mathcal{V}_i \)’s.
- A quantum state, a vector in \( \mathcal{V}_{tot} \):
  \( |\psi\rangle = \sum \psi(m_1, ..., m_N) |m_1\rangle \otimes ... \otimes |m_N\rangle \),
  \( |m_i\rangle \in \mathcal{V}_i \)
- A gapped Hamiltonian has following spectrum as \( N \to \infty \) (eg \( H = -\sum (J\sigma_i^z\sigma_{i+\delta}^z + h\sigma_i^x) \))
- A particle-like excitation:
  energy density \( = \langle \psi | H_x | \psi \rangle \)
- Quasiparticle energy gap \( \Delta_p \)
  - How to measure \( \Delta \) and \( \Delta_p \)?

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Consider a system formed by three spin-1/2 spins. The Hamiltonian of the system is given by

\[ H = -\sigma_1^z \sigma_2^z - \sigma_2^z \sigma_3^z + h(\sigma_1^x + \sigma_2^x + \sigma_3^x) \]

where \( \sigma_i^{x,y,z} \) are the Pauli matrices acting on the \( i^{th} \) spin. (The above is a transverse field Ising model). \( H \) is an 8-by-8 matrix. Write down such a 8-by-8 matrix.

Some further questions to think about:

- If we view down-spin as vacuum and up-spin as a boson, we can view a hard-core boson system as a spin-1/2 system. Now we view a system of hard bosons hopping on a ring of \( L \) sites as a spin-1/2 system. How to write down the spin Hamiltonian to describe such a boson-hopping system?

- Using the same picture, we can also view a system of spin-less fermions hopping on a ring of \( L \) sites as a spin-1/2 system. How to write down the spin Hamiltonian to describe such a fermion-hopping system?
Phases are defined through phase transitions.

**What are phase transitions?**

As we change a parameter $g$ in Hamiltonian $H(g)$, the ground state energy density $\epsilon_g = E_g / V$ or average of some other local operators $\langle \hat{Q} \rangle$ may have a singularity at $g_c \rightarrow$ the system has a phase transition at $g_c$. The Hamiltonian $H(g)$ is a smooth function of $g$. How can the ground state energy density $\epsilon_g$ be singular at a certain $g_c$?

Spontaneous symmetry breaking is a mechanism to cause a singularity in ground state energy density $\epsilon_g$. $\rightarrow$ Spontaneous symmetry breaking causes phase transition.
Symmetry breaking theory of phase transition

- Ground state is obtained by minimizing an energy density function $\epsilon_g(\phi)$ against the internal variable $\phi$.
  $\epsilon_g(\phi)$ is a smooth function of $\phi$ and $g$. How can its minimal value $\epsilon_g \equiv \epsilon_g(\phi_{\text{min}})$ have singularity as a function of $g$?  
- Minimum splitting $\rightarrow$ singularity in ground state energy $\epsilon_g$ at $g_c$.
  State-B has less symmetry than state-A.
  State-A $\rightarrow$ State-B: spontaneous symmetry breaking.
- For a long time, we believe that 
  $\textit{phase transition} = \textit{change of symmetry}$
  $\textit{the different phases} = \textit{different symmetry}$.  

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Long-range entangled quantum matter and a unification of forces
Example: transverse-field Ising model

- Hamiltonian \( H = - \sum (J \sigma_i^z \sigma_{i+\delta}^z + h \sigma_i^x) \)
- Trial ground state wave function, a product state:
  \[
  |\Psi\rangle = \bigotimes_i |\psi_i\rangle \\
  |\psi_i\rangle = \cos(\phi/2) |\uparrow\rangle + \sin(\phi/2) |\downarrow\rangle
  \]

- A symmetry breaking phase transition happens at \( h/J = \eta_c \), where
  - Ground state energy density \( \epsilon_{J,h} = \min[\epsilon_{J,h}(\phi)] \) has a singularity
  - \( \Delta, \Delta_p \) also have singularities, and vanish at the transition
  - Every physical quantities have singularities at the transition
- **The math foundation is group theory:** classified by \((G_H, G_\Psi)\)
  From 230 ways of translation symmetry breaking, we obtain the 230 crystal orders in 3D.
Consider a system formed by \( L \) spin-1/2 spins. The Hamiltonian of the system is given by 
\[
H = -\sum_{i=1}^{L-1} \sigma_i^z \sigma_{i+1}^z + h \sum_{i=1}^{L} \sigma_i^x,
\]
which has a \( Z_2 \) symmetry. Here \( \sigma_i^{x,y,z} \) are the Pauli matrices acting on the \( i^{th} \) spin. (The above is a transverse field Ising model). \( H \) is an \( 2^L \)-by-\( 2^L \) matrix, whose eigenvalues can be computed via the following octave code (the code may also run in matlab):

```octave
X=sparse([0,1;1,0]); Z=sparse([1,0;0,-1]); ZZ=kron(Z,Z);
L=12
h=0.1
H=sparse(2^L,2^L);
for i=1:L-1
    H=H + kron( speye(2^(i-1)), kron( ZZ, speye(2^(L-1-i))) ) ;
end
for i=1:L
    H=H + kron( speye(2^(i-1)), kron( h*X, speye(2^(L-i))) ) ;
end
# compute the lowest 10 eigenvalues
eigs( H , 10, 'sa')
```

*Use octave or matlab to compute the 10 lowest energy eigenvalues for \( L = 12 \) and plot them as a function of \( h \). Guess the critical value of \( h \) where there is a symmetry breaking transition. What is the \( Z_2 \) symmetry?*
Consider a transverse field Ising model \( H = - \sum \sigma^z_i \sigma^z_{i+1} + h \sum \sigma^x_i \).

*Use the following trial wave function*

\[
|\psi\rangle = \bigotimes_i |\psi_i\rangle = \cos(\phi/2)|\uparrow\rangle + \sin(\phi/2)|\downarrow\rangle
\]

*to estimate the ground state energy by computing \( \langle \Psi | H | \Psi \rangle \). Plot the estimated ground state energy as a function of \( h \) and find the critical \( h \) where there is a (mean-field) symmetry breaking transition*

**Some further questions to think about:**

- Can we find a mean-field phase transition as we change \( h \) in a similar model with \( U(1) \) symmetry

\[
H = - \sum_{i=1}^{L-1} [\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1}] + h \sum_{i=1}^L \sigma^z_i
\]
A quantum picture of symmetry breaking order long range correlation and Greenberger-Horne-Zeilinger (GHZ) entanglement

But the true ground of $H$ with symmetry $H = U^\dagger(g)HU(g)$ also have the symmetry $e^{i\theta(g)}\ket{\psi_0} = U(g)\ket{\psi_0}$

- How to detect symmetry breaking phase within quantum theory?
  1. degenerate space with different symmetry quantum numbers
  2. long range correlation
  3. GHZ entanglement

For a long time, we thought Landau symmetry breaking theory describes all the phases and phase transitions.
In graduate study after 1980’s, we learned ...

there are much more than symmetry-breaking phases:

- **Example:**
  - Quantum Hall states \( \sigma_{xy} = \frac{m}{n} \frac{e^2}{h} \)
  - Spin liquid states

- FQH states and spin-liquid states have have different phases with no symmetry breaking, no crystal order, no spin order, ... so they must have a new order – **topological order** Wen 89
What IS topological order?

To define a physical concept, such as symmetry-breaking order or topological order, is to design a probe to measure it.

For example,
- crystal order is defined/probed by X-ray diffraction:

![X-ray diffraction diagram](image-url)
Symmetry-breaking orders through experiments

<table>
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- All the above probes are linear responses. But topological order cannot be probed/defined through linear responses.
Topological order can be defined “experimentally” through two unusual topological probes (at least in 2D)

(1) **Topology-dependent ground state degeneracy** \( D_g \) Wen 89

\[
\begin{align*}
g=0 & : \text{Deg.}=1 \\
g=1 & : \text{Deg.}=D_1 \\
g=2 & : \text{Deg.}=D_2
\end{align*}
\]

(2) **Non-Abelian geometric’s phases** of the degenerate ground state from deforming the torus: Wen 90

- Shear deformation \( T \): \( |\Psi_\alpha\rangle \rightarrow |\Psi'_\alpha\rangle = T_{\alpha\beta} |\Psi_\beta\rangle \)

- 90° rotation \( S \): \( |\Psi_\alpha\rangle \rightarrow |\Psi''_\alpha\rangle = S_{\alpha\beta} |\Psi_\beta\rangle \)

- \( T, S \), define topological order “experimentally”.
- \( T, S \) is a *universal probe* for any 2D topological orders, just like X-ray is a universal probe for any crystal orders.
Real experimental probe of topological orders: No finite-temperature phase transition (why it is a probe?)

Herbertsmithsite: spin-1/2 on Kagome lattice

\[ \mathcal{H} = J \sum_i S_i \cdot S_j \]

\( J \sim 200 \) K, no phase trans. down to 50 mK \( \rightarrow \) spin liquid

Helton et al.

Numerical calculations Misguich-Bernu-Lhuillier-Waldtmann 98; Jiang-Weng-Sheng 08; Yan-Huse-White 10 \( \rightarrow \) \( \mathbb{Z}_2 \) topological order

Read-Sachdev 91, Wen 91

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Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
Real experimental probe of topological orders: Robust gapless boundary excitations against any perturbations

- Chiral edge transport \( \text{Halperin 85 for IQH, Wen 89 for FQH} \rightarrow \text{perfect conductor} \)
- Different bulk topological orders \( \rightarrow \) different edge states \( \text{Wen 89} \)
- Abelian FQH states with \( \nu = 1/3, 2/3, 4/3, 5/3, \ldots \) have an integral number of edge branches. Non-Abelian FQH states with \( \nu = 5/2, \ldots \) have a factional number of edge branches
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Long-range entangled quantum matter and a unification of forces
Real experimental probe of topological orders: fractional charge/quantum-number and fractional statistics

Hubbard model on triangular lattice:

\[ t'/t = 0.5 \sim 1.1 \]

\[ X=Cu[N(CN)_2]Cl, \text{ Cu}_2(CN)_3,... \]

Spin interaction \( J = 250K \)
But no AF order down to \( 35mK \)

\[ \text{Cu}[N(CN)_2]Cl \ t'/t = .75 \] \[ \text{Cu}_2(CN)_3 \ t'/t = 1.06 \]

- Spin-charge separation + emergent fermion \( \rightarrow \) spinon Fermi surface

\[ \text{Xiao-Gang Wen} \]

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- The linear-response probe **Zero-resistance** and **Meissner effect** define **superconducting order**. Treating the EM fields as non-dynamical fields

- The topological probe **Topological degeneracy** and **non-Abelian geometric phases** \(T, S\) define a completely new class of order – **topological order**.

- \(T, S\) determines the quasiparticle statistics. Keski-Vakkuri & Wen 93; Zhang-Grover-Turner-Oshikawa-Vishwanath 12; Cincio-Vidal 12
What is the microscopic picture of topological order?

But what is the microscopic understanding of topological order?

Zero-resistance and Meissner effect → experimental definition of superconducting order.

It took 40 years to gain a microscopic picture of superconducting order: electron-pair condensation

Bardeen-Cooper-Schrieffer 57
What is the microscopic picture of topological order?

- But what is the microscopic understanding of topological order?
- Zero-resistance and Meissner effect $\rightarrow$ experimental definition of superconducting order.
- It took 40 years to gain a microscopic picture of superconducting order: \textit{electron-pair condensation} Bardeen-Cooper-Schrieffer 57
- It took 20 years to gain a microscopic understanding of topological order: \textit{long-range entanglements} Chen-Gu-Wen 10 (defined by local unitary trans. and motivated by topological entanglement entropy). Kitaev-Preskill 06, Levin-Wen 06
Quantum entanglements through examples

• $|\uparrow\rangle \otimes |\downarrow\rangle = \text{direct-product state} \rightarrow \text{unentangled (classical)}$
Quantum entanglements through examples

- $|↑⟩ ⊗ |↓⟩ = \text{direct-product state} \rightarrow \text{unentangled (classical)}$
- $|↑⟩ ⊗ |↓⟩ + |↓⟩ ⊗ |↑⟩ \rightarrow \text{entangled (quantum)}$
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- $|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \rightarrow \text{more entangled}$

Long-range entangled quantum matter and a unification of forces
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- |↑⟩ ⊗ |↓⟩ + |↓⟩ ⊗ |↑⟩ → entangled (quantum)
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$= (|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle) = |x\rangle \otimes |x\rangle \rightarrow \text{unentangled}$

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  \(= (|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle) = |x\rangle \otimes |x\rangle \rightarrow \text{unentangled}\)
- \(\otimes\) \(\otimes\) \(\rightarrow \text{unentangled}\)
- \((|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes \rightarrow \text{short-range entangled (SRE) entangled}\)
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  $= (|↑⟩ + |↓⟩) \otimes (|↑⟩ + |↓⟩) = |x⟩ \otimes |x⟩ \rightarrow \text{unentangled}$
- $\ldots \rightarrow \text{unentangled}$
- $\ldots \rightarrow \text{short-range entangled (SRE) entangled}$

- Crystal order: $|Φ_{\text{crystal}}⟩ = |\cdots⟩ = |0⟩_{x_1} \otimes |1⟩_{x_2} \otimes |0⟩_{x_3} \ldots$
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- Crystal order: $|Φ_{\text{crystal}}⟩ = \quad \rightarrow \text{direct-product state} \rightarrow \text{unentangled state (classical)}$
- Particle condensation (superfluid)
  
  $|Φ_{\text{SF}}⟩ = \sum_{\text{all conf.}} |\begin{array}{|c|}
  \hline
  \cdot & \cdot & \cdot \\
  \hline
\end{array}| = |0⟩_{x_1} \otimes |1⟩_{x_2} \otimes |0⟩_{x_3} ...$
Quantum entanglements through examples

- $|\uparrow\rangle \otimes |\downarrow\rangle = \text{direct-product state} \rightarrow \text{unentangled (classical)}$
- $|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \rightarrow \text{entangled (quantum)}$
- $|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle = (|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle) = |x\rangle \otimes |x\rangle \rightarrow \text{unentangled}$
- $\bigotimes \rightarrow \text{unentangled}$
- $\bigotimes \rightarrow \text{short-range entangled (SRE) entangled}$

- Crystal order: $|\Phi_{\text{crystal}}\rangle = \bigotimes \rightarrow \text{direct-product state} \rightarrow \text{unentangled state (classical)}$

- Particle condensation (superfluid)
  
  $|\Phi_{\text{SF}}\rangle = \sum_{\text{all conf.}} |\bigotimes\rangle = (|0\rangle_{x_1} + |1\rangle_{x_1} + \ldots) \otimes (|0\rangle_{x_2} + |1\rangle_{x_2} + \ldots) \ldots$

  $\rightarrow \text{direct-product state} \rightarrow \text{unentangled state (classical)}$

- Superfluid, as an exemplary quantum state of matter, is actually very classical and unquantum from entanglement point of view.
How to make long range entanglements (topo. orders)

• **First example** sum over a subset of the particle configurations by first join the particles into strings, then sum over the loop states

\[ |\Phi_{\text{loop}}\rangle = \sum_{\text{all loop conf.}} \left| \begin{array}{c} \vdots \\ \end{array} \right| \]

→ string-net condensation (string liquid):

Kogut-Susskind 75, Kitaev 97, Wen 03, Levin-Wen 05

• **Second example** scamble the phases Laughlin 83

\[ \Psi_{\text{FQH}}(z_1, z_2, \cdots) = \prod (z_i - z_j) e^{-\frac{1}{4} \sum |z_i|^2}, \quad \Psi_{\text{SF}}(z_1, z_2, \cdots) = 1. \]

• How to quantitatively describe those long range entanglements? How fractional charge, fractional/Fermi statistics, gauge interaction emerge from long range
Topo-order/Long-range-entanglement through pictures

Symmetry breaking orders through pictures

Ferromagnet Anti-ferromagnet

Superfluid of bosons Superconductor of fermions

• every particle is doing its own dancing (relative angular momentum → s-wave, d-wave,...),
• every particle is doing the same dancing → Long-range order

Xiao-Gang Wen

Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
Symmetry breaking orders through pictures

Ferromagnet

Anti-ferromagnet

Superfluid of bosons

Superconductor of fermions

- every particle is doing its own dancing (relative angular momentum $\rightarrow$ s-wave, d-wave,...),
- every particle is doing the same dancing $\rightarrow$ **Long-range order**
Global dance:
All spins/particles dance following a local dancing “rules”
→ The spins/particles dance collectively
→ a global dancing pattern.

Topological physical properties:
Using the quantitative description of the local dancing “rules”
→ Topological physical properties of topologically ordered states.
Local dancing rule $\rightarrow$ global dancing pattern

- Local dancing rules of a FQH liquid:
  1. every electron dances around clock-wise
     $(\Phi_{\text{FQH}}$ only depends on $z = x + iy$)
  2. takes exactly three steps to go around any others
     $(\Phi_{\text{FQH}}$'s phase change $6\pi$)
→ Global dancing pattern $\Phi_{\text{FQH}}(\{z_1, \ldots, z_N\}) = \prod(z_i - z_j)^3$

- A systematic theory of FQH state – Pattern of zeros $S_a$:
a-electron cluster has a relative angular momentum $S_a$ Wen-Wang 08

- Local dancing rules are enforce by the Hamiltonian to lower energy.
- Only certain sequences $S_a$ correspond to valid FQH states. Which?
- Different POZ $S_a$ give rise to different topological properties.
Introduction of fractional quantum Hall (FQH) states

- One-particle in magnetic field (choose $B = 1$ and $z = x + iy$):
  \[ H_0 = - \sum (\partial_z - \frac{B}{4} z^*)(\partial_{z^*} + \frac{B}{4} z) \]

  Lowest energy eigenstates: $P(z) e^{-\frac{1}{4}|z|^2}$, $P(z) = \sum a_l z^l$

  since $e^\frac{1}{4}zz^*(i\partial_z - i\frac{1}{4}z^*)(i\partial_{z^*} + i\frac{1}{4}z)e^{-\frac{1}{4}zz^*} = (i\partial_z - i\frac{1}{2}z^*)i\partial_{z^*}$
Introduction of fractional quantum Hall (FQH) states

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- $N$-electrons (fermionic or bosonic) in a magnetic field:
  
  $H(g_1, g_2) = \sum_{i=1}^N (i\partial_{z_i} - i\frac{B}{4} z_i^*)(i\partial_{z_i^*} + i\frac{B}{4} z_i) + \sum_{i<j} V_{g_1,g_2}(x_i - x_j, y_i - y_j)$

- When $V_{g_1,g_2} = 0$, there are many minimal energy wave functions

  $\Psi = P(z_1, \ldots , z_N)e^{-\frac{1}{4} \sum_{i=1}^N z_i z_i^*}$, $P = a$ (anti-)symmetric polynomial

  all have zero energy (for any $P$):

  $\left[ \sum_{i=1}^N (i\partial_{z_i} - i\frac{B}{4} z_i^*)(i\partial_{z_i^*} + i\frac{B}{4} z_i) \right] P(z_1, \ldots , z_N)e^{-\frac{1}{4} \sum_{i=1}^N z_i z_i^*} = 0$
Introduction of fractional quantum Hall (FQH) states

- One-particle in magnetic field (choose $B = 1$ and $z = x + iy$):
  \[ H_0 = -\sum (\partial z - \frac{B}{4} z^*)(\partial z^* + \frac{B}{4} z) \]
  Lowest energy eigenstates:
  \[ P(z)e^{-\frac{1}{4}|z|^2}, \quad P(z) = \sum a_l z^l \]
  since
  \[ e^{\frac{1}{4}zz^*}(i\partial z - i\frac{1}{4} z^*)(i\partial z^* + i\frac{1}{4} z)e^{-\frac{1}{4}zz^*} = (i\partial z - i\frac{1}{2} z^*)i\partial z^* \]

- $N$-electrons (fermionic or bosonic) in a magnetic field:
  \[ H(g_1, g_2) = \sum_{i=1}^{N} (i\partial z_i - i\frac{B}{4} z_i^*)(i\partial z_i^* + i\frac{B}{4} z_i) + \sum_{i<j} V_{g_1,g_2}(x_i - x_j, y_i - y_j) \]

- When $V_{g_1,g_2} = 0$, there are many minimal energy wave functions
  \[ \Psi = P(z_1, \ldots, z_N)e^{-\frac{1}{4}\sum_{i=1}^{N} z_i z_i^*}, \quad P = \text{a (anti-)symmetric polynomial} \]
  all have zero energy (for any $P$):
  \[ \left[ \sum_{i=1}^{N} (i\partial z_i - i\frac{B}{4} z_i^*)(i\partial z_i^* + i\frac{B}{4} z_i) \right] P(z_1, \ldots, z_N)e^{-\frac{1}{4}\sum_{i=1}^{N} z_i z_i^*} = 0 \]

- For small non-zero $V_{g_1,g_2}$, there is only one minimal energy wave function $P$ whose form is determined by $V_{g_1,g_2}$.
Show that the Hamiltonian for a 2D electron in a uniform magnetic field $B$ is given by

$$H_0 = - \sum \left( \partial_z - \frac{B}{4} z^* \right) \left( \partial_{z^*} + \frac{B}{4} z \right) + \text{const}. $$

in complex coordinate $z = x + iy$ (in $\hbar = c = e = 1$ unit). Find the constant term $\text{const}$. 
3 ideal FQH states: the exact zero-energy ground states

- $\nu = 1/2$ bosonic Laughlin state: $z_1 \approx z_2$, second-order zero
  \[ P_{1/2} = \prod_{i<j} (z_i - z_j)^2, \quad V_{1/2}(z_1, z_2) = \delta(z_1 - z_2), \]
  \[ \left[ \sum_{i<j} V_{1/2}(z_i - z_j) \right] P_{1/2} = 0. \]

All other states have finite energies in $N \to \infty$ limit (gapped).
3 ideal FQH states: the exact zero-energy ground states

• $\nu = 1/2$ bosonic Laughlin state: $z_1 \approx z_2$, second-order zero

$$P_{1/2} = \prod_{i<j} (z_i - z_j)^2, \quad V_{1/2}(z_1, z_2) = \delta(z_1 - z_2),$$

$$[\sum_{i<j} V_{1/2}(z_i - z_j)] P_{1/2} = 0.$$  

All other states have finite energies in $N \to \infty$ limit (gapped).

• $\nu = 1/4$ bosonic Laughlin state: $z_1 \approx z_2$, fourth-order zero

$$P_{1/4} = \prod_{i<j} (z_i - z_j)^4,$$

$$V_{1/4}(z_1, z_2) = v_0 \delta(z_1 - z_2) + v_2 \partial_{z_1}^2 \delta(z_1 - z_2) \partial_{z_1}^2.$$
3 ideal FQH states: the exact zero-energy ground states

- $\nu = 1/2$ bosonic Laughlin state: $z_1 \approx z_2$, second-order zero
  \[ P_{1/2} = \prod_{i<j} (z_i - z_j)^2, \quad V_{1/2}(z_1, z_2) = \delta(z_1 - z_2), \]
  \[
  \left[ \sum_{i<j} V_{1/2}(z_i - z_j) \right] P_{1/2} = 0.
  \]
  All other states have finite energies in $N \to \infty$ limit (gapped).

- $\nu = 1/4$ bosonic Laughlin state: $z_1 \approx z_2$, fourth-order zero
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- $\nu = 1$ Pfaffian state: $z_1 \approx z_2$, no zero; $z_1 \approx z_2 \approx z_3$, second-order zero;
  \[ P_{Pf} = A \left( \frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots \frac{1}{z_{N-1} - z_N} \right) \prod_{i<j} (z_i - z_j) = \text{Pf} \left( \frac{1}{z_i - z_j} \right) \prod_{i<j} (z_i - z_j) \]
  \[ V_{Pf}(z_1, z_2, z_3) = S \left[ v_0 \delta(z_1 - z_2) \delta(z_2 - z_3) - v_1 \delta(z_1 - z_2) \partial_{z_3}^* \delta(z_2 - z_3) \partial_{z_3} \right] \]
3 ideal FQH states: the exact zero-energy ground states

- \( \nu = 1/2 \) bosonic Laughlin state: \( z_1 \approx z_2 \), second-order zero
  \[
  P_{1/2} = \prod_{i<j}(z_i - z_j)^2, \quad V_{1/2}(z_1, z_2) = \delta(z_1 - z_2),
  \]
  \[
  \left[ \sum_{i<j} V_{1/2}(z_i - z_j) \right] P_{1/2} = 0.
  \]
  All other states have finite energies in \( N \to \infty \) limit (gapped).

- \( \nu = 1/4 \) bosonic Laughlin state: \( z_1 \approx z_2 \), fourth-order zero
  \[
  P_{1/4} = \prod_{i<j}(z_i - z_j)^4
  \]
  \[
  V_{1/4}(z_1, z_2) = v_0 \delta(z_1 - z_2) + v_2 \partial_z^2 \partial_{z_1}^2 \delta(z_1 - z_2)\partial_z^2
  \]

- \( \nu = 1 \) Pfaffian state: \( z_1 \approx z_2 \), no zero; \( z_1 \approx z_2 \approx z_3 \), second-order zero;
  \[
  P_{Pf} = A\left(\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots \frac{1}{z_{N-1} - z_N}\right) \prod_{i<j}(z_i - z_j) = Pf\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}(z_i - z_j)
  \]
  \[
  V_{Pf}(z_1, z_2, z_3) = S[v_0 \delta(z_1 - z_2)\delta(z_2 - z_3) - v_1 \delta(z_1 - z_2)\partial_{z_3}^2 \delta(z_2 - z_3)\partial_{z_3}]
  \]

- \( \nu = 1 \) fermionic IQH state: \( z_1 \approx z_2 \), first-order zero;
  \[
  P_{Pf} = \prod_{i<j}(z_i - z_j), \quad V_{Pf}(z_1, z_2) = 0
  \]
Pattern of zeros – dancing pattern for FQH states

Let \( z_i = \lambda \xi_i + z^{(a)}, \ i = 1, 2, \cdots, a \)

\[
P(\{z_i\}) = \lambda^{S_a} \tilde{P}(\xi^1, \ldots, \xi^a; z^{(a)}, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_a+1})
\]

- The sequence of integers \( \{S_a\} \) characterizes the polynomial \( P(\{z_i\}) \) and is called the pattern of zeros.
- \( S_a \) is the relative angular momentum of an \( a \)-electron cluster.

\[
\psi_1 = \prod_{i<j} (z_i - z_j) \rightarrow S_1, S_2, = 0, 1, 3, 6, 10, \ldots
\]

\[
\psi_{1/2} = \prod_{i<j} (z_i - z_j)^2 \rightarrow S_1, S_2, = 0, 2, 6, 12, 20, \ldots
\]

\[
\psi_{1/4} = \prod_{i<j} (z_i - z_j)^4 \rightarrow S_1, S_2, = 0, 4, 12, 24, 40, \ldots
\]

\[
P_{Pf} = A\left(\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots\right) \prod_{i<j} (z_i - z_j) \rightarrow S_1, S_2, = 0, 0, 2, 4, 8, 12, 18, 24,
\]
The physical meaning of pattern of zeros: pattern of zeros and orbital distribution

$S_N$ is the total power of $z_i$ if the polynomial has $N$ variables.
Relation to monomial symmetric polynomial ("highest weight"):
Let $l_a = S_a - S_{a-1}$ or $S_a = \sum_{i=1}^{a} l_i$, then

$$P(\{z_i\}) \sim S[z_1^{l_1}z_2^{l_2}\cdots] + \cdots, \quad = m_{\{l_1,l_2,\ldots\}}(z_i,\ldots,z_N) + \cdots$$

The monomial symmetric polynomial can produce any sequences $S_a$ if we treat $S_a$ as the POZ at $z = 0$.

- **Pattern of zeros and occupation distribution**
  Convert $l_a$ to $n_l$, the occupation distribution: $n_l = \sum_{a=1}^{\infty} \delta_{l_a,l}$
\[ z^l e^{-\frac{1}{4} |z|^2}, \quad \Psi = \prod_{i<j} (z_i - z_j) e^{-\frac{1}{4} \sum |z_i|^2} = \mathcal{A}[(z_1)^0(z_2)^1 \ldots] e^{-\frac{1}{4} \sum |z_i|^2} \]

\[ \prod_{i<j} (z_i - z_j) = \mathcal{A}[(z_1)^0(z_2)^1 \ldots] \rightarrow S_1, S_2, \ldots = 0, 1, 3, 6, 10, \ldots \]

\[ l_1, l_2, \ldots : 0, 1, 2, 3, 4, \ldots , \quad n_0 n_1 n_2 \ldots : 111111111111 \ldots \]

For \( P_{1/2} \):

\[ \prod_{i<j} (z_i - z_j)^2 = \left( \mathcal{A}[(z_1)^0(z_2)^1(z_3)^2 \ldots] \right)^2 \rightarrow S_1, S_2, \ldots = 0, 2, 6, 12, 20, \ldots \]

\[ l_1, l_2, \ldots : 0, 2, 4, 6, 10, \ldots , \quad n_0 n_1 n_2 \ldots : 10101010101010 \ldots \]

root monomial poly. \( = S[(z_1)^0(z_2)^2(z_3)^4 \ldots] \)
\[ \nu = \frac{1}{4} \text{ Laughlin state } P_{1/4} = \prod_{i<j}(z_i - z_j)^2 \]

\[ S_1, S_2, \ldots : 0, 4, 12, 24, 40, 60, 84, \ldots \]

\[ l_1, l_2, \ldots : 0, 4, 8, 12, 16, 20, \ldots \]

\[ n_0 n_1 n_2 \ldots : 100010001000100010001 \ldots \]

root monomial poly. \[ = S[(z_1)^0(z_2)^4(z_3)^8\ldots] \]

A cluster (unit cell): 1 particles 4 orbitals

\[ \nu = 1 \text{ Pfaffian state } P_{Pf} = A\left( \frac{1}{z_1-z_2} \frac{1}{z_3-z_4} \ldots \right) \prod_{i<j}(z_i - z_j) \]

\[ S_1, S_2, \ldots : 0, 0, 2, 4, 8, 12, 18, 24, \ldots \]

\[ l_1, l_2, \ldots : 0, 0, 2, 2, 4, 4, 6, 6, \ldots \]

\[ n_0 n_1 n_2 \ldots : 202020202020202020202 \ldots \]

root monomial poly. \[ = S[(z_1)^0(z_2)^0(z_3)^2(z_4)^2(z_5)^4(z_6)^4\ldots] \]

A cluster (unit cell): 2 particles 2 orbitals
• \( \nu = 1/4 \) Laughlin state \( P_{1/4} = \prod_{i<j} (z_i - z_j)^2 \)

\[
S_1, S_2, \cdots : 0, 4, 12, 24, 40, 60, 84, \cdots \\
l_1, l_2, \cdots : 0, 4, 8, 12, 16, 20, \cdots \\
n_0 n_1 n_2 \cdots : 100010001000100010001 \cdots \\
\text{root monomial poly. } \ = S[(z_1)^0(z_2)^4(z_3)^8\cdots]
\]

A cluster (unit cell): 1 particles 4 orbitals

• \( \nu = 1 \) Pfaffian state \( P_{Pf} = A\left(\frac{1}{z_1-z_2} \frac{1}{z_3-z_4} \cdots \right) \prod_{i<j} (z_i - z_j) \)

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S_1, S_2, \cdots : 0, 0, 2, 4, 8, 12, 18, 24, \cdots \\
l_1, l_2, \cdots : 0, 0, 2, 2, 4, 4, 6, 6, \cdots \\
n_0 n_1 n_2 \cdots : 20202020202020202020202020202020 \cdots \\
\text{root monomial poly. } \ = S[(z_1)^0(z_2)^0(z_3)^2(z_4)^2(z_5)^4(z_6)^4\cdots]
\]

A cluster (unit cell): 2 particles 2 orbitals

• FQH \( \leftrightarrow \) 1D pattern (by considering FQH state on thin cylinder)

Haldane & Rezayi, 94; Seidel & Lee, 06; Bergholtz, Kailasvuori, Wikberg, Hansson, Karlhede, 06; Bernevig & Haldane, 07

Xiao-Gang Wen  Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
Find the pattern of zeros $S_a$ (and $I_a$, $n_l$) for the following wave functions

(1) $\Psi_1 = S\left(\frac{1}{(z_1 - z_2)^2} \frac{1}{(z_3 - z_4)^2} \cdots \right) \prod_{i<j}(z_i - z_j)^2$ where $S$ is a symmetrization operator.

(2) $\Psi_2 = [A\left(\frac{1}{(z_1 - z_2)} \frac{1}{(z_3 - z_4)} \cdots \right)]^2 \prod_{i<j}(z_i - z_j)^2$ where $A$ is an anti-symmetrization operator.

(3) $\Psi_3 = A\left(\frac{1}{(z_1 - z_2)^3} \frac{1}{(z_3 - z_4)^3} \cdots \right) \prod_{i<j}(z_i - z_j)^3$

(4) $\Psi_4 = [A\left(\frac{1}{(z_1 - z_2)} \frac{1}{(z_3 - z_4)} \cdots \right)]^3 \prod_{i<j}(z_i - z_j)^3$

Are $\Psi_1$ and $\Psi_2$ the same wave function? Are $\Psi_3$ and $\Psi_4$ the same wave function? (You may use mathematica etc)
A classification problem

- We have seen that each symmetric polynomial $P(\{z_i\}) \rightarrow \{S_a\}$ a pattern of zeros. But each sequence of integers $\{S_a\} \not\rightarrow P(\{z_i\})$

- Find all the conditions a sequence $\{S_a\}$ must satisfy, such that $\{S_a\}$ describe a symmetric polynomial that satisfies the unique fusion condition. →

A classification of symmetric polynomials (FQH states) through pattern of zeros.
Derived polynomials

- **Unique fusion cond.**: Let \( z_i = \lambda \xi_i + z^{(a)}, \ i = 1, 2, \cdots, a \)

\[
P(\{z_i\}) = \lambda S_a \tilde{P}(\xi^1, \ldots, \xi^a; z^{(a)}, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_a+1})
\]

\( \tilde{P} \) does not depend on the “shape” \( \{\xi^i\} \)

\[
\tilde{P}(\{\xi^i\}; z^{(a)}, z_{a+1}, z_{a+2}, \cdots) = F(\{\xi^i\}) P_{\text{derived}}(z^{(a)}, z_{a+1}, z_{a+2}, \cdots)
\]

- So, we can rewrite the above as

\[
P(\{z_i\}) = \lambda S_a P_{\text{derived}}(z^{(a)}, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_a+1})
\]

- If we repeat the above, we get a derived polynomial

\[
P_{\text{derived}}(z^{(a)}, z^{(b)}, z^{(c)}, \cdots).
\]
Derived polynomials

- **Unique fusion cond.**: Let \( z_i = \lambda \xi_i + z^{(a)} \), \( i = 1, 2, \cdots, a \)

\[
P(\{z_i\}) = \lambda S_a \tilde{P}(\xi^1, \ldots, \xi^a; z^{(a)}, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_a+1}),
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\]

- If we repeat the above, we get a derived polynomial

\[
P_{\text{derived}}(z^{(a)}, z^{(b)}, z^{(c)}, \cdots).
\]

- Zeros in derived polynomials \( D_{a,b} \)

\[
P_{\text{derived}}(z^{(a)}, z^{(b)}, z^{(c)}, \cdots) \sim (z^{(a)} - z^{(b)}) D_{a,b} P'_{\text{derived}}(z^{(a+b)}, \cdots) + \cdots
\]

also characterize the pattern of zeros.
**Derived polynomials**

- **Unique fusion cond.**: Let \( z_i = \lambda \xi_i + z^{(a)}, \ i = 1, 2, \ldots, a \)

\[
P(\{z_i\}) = \lambda^S a \tilde{P}(\xi^1, \ldots, \xi^a; z^{(a)}, z_{a+1}, z_{a+2}, \ldots) + O(\lambda^{S_a+1}),
\]

\( \tilde{P} \) does not depend on the “shape” \( \{\xi^i\} \)

\[
\tilde{P}(\{\xi^i\}; z^{(a)}, z_{a+1}, z_{a+2}, \ldots) = F(\{\xi^i\})P_{\text{derived}}(z^{(a)}, z_{a+1}, z_{a+2}, \ldots)
\]

- So, we can rewrite the above as

\[
P(\{z_i\}) = \lambda^S a P_{\text{derived}}(z^{(a)}, z_{a+1}, z_{a+2}, \ldots) + O(\lambda^{S_a+1})
\]

- If we repeat the above, we get a derived polynomial

\[
P_{\text{derived}}(z^{(a)}, z^{(b)}, z^{(c)}, \ldots).
\]

- Zeros in derived polynomials \( D_{a,b} \)

\[
P_{\text{derived}}(z^{(a)}, z^{(b)}, z^{(c)}, \ldots) \sim (z^{(a)} - z^{(b)}) D_{a,b} P'_{\text{derived}}(z^{(a+b)} \ldots) + \ldots
\]

also characterize the pattern of zeros.

- The data \( D_{a,b} \) and \( S_a \) are one-to-one related:
Conditions on pattern of zeros – ground state

- **Concave conditions**

\[ \Delta_2(a, b) \equiv S_{a+b} - S_a - S_b = D_{a,b} \geq 0, \]

The second one comes from \[ D_{a+b}, c \geq D_{a, c} + D_{b, c} \] which can be shown by considering \( P \) derived (\( z(a), z(b), z(c), \ldots \)) as a function of \( z(c) \).
Conditions on pattern of zeros – ground state

- **Concave conditions**

\[
\Delta_2(a, b) \equiv S_{a+b} - S_a - S_b = D_{a,b} \geq 0,
\]

\[
\Delta_3(a, b, c) \equiv S_{a+b+c} - S_{a+b} - S_{b+c} - S_{a+c} + S_a + S_b + S_c \geq 0
\]
Conditions on pattern of zeros – ground state

- **Concave conditions**

\[
\Delta_2(a, b) \equiv S_{a+b} - S_a - S_b = D_{a,b} \geq 0,
\]

\[
\Delta_3(a, b, c) \equiv S_{a+b+c} - S_{a+b} - S_{b+c} - S_{a+c} + S_a + S_b + S_c \geq 0
\]

The second one comes from

\[
D_{a+b,c} \geq D_{a,c} + D_{b,c}
\]

which can be shown by considering \( P_{\text{derived}}(z^{(a)}, z^{(b)}, z^{(c)}, \ldots) \) as a function of \( z^{(c)} \).
• **$n$-cluster condition**: No off-particle zeros when $c = n$ (or the wave function for the $n$-clusters is the Laughlin wave function or a $n$-electron cluster is “kind of trivial”)

\[ D_{a+b,n} = D_{a,n} + D_{b,n} \]

\[ S_{a+kn} = S_a + kS_n + \frac{k(k-1)nm}{2} + kma \]
• **n-cluster condition**: No off-particle zeros when \( c = n \) (or the wave function for the \( n \)-clusters is the Laughlin wave function or a \( n \)-electron cluster is “kind of trivial”)

\[
D_{a+b,n} = D_{a,n} + D_{b,n} \rightarrow
S_{a+kn} = S_a + kS_n + \frac{k(k-1)nm}{2} + kma
\]

Since \( S_1 = 0 \), \((m, S_2, \cdots, S_n)\) carries all the information about the pattern of zeros from an \( n \)-cluster symmetric polynomial.
• **n-cluster condition**: No off-particle zeros when \( c = n \) (or the wave function for the \( n \)-clusters is the Laughlin wave function or a \( n \)-electron cluster is “kind of trivial”)

\[
\rightarrow D_{a+b,n} = D_{a,n} + D_{b,n} \rightarrow
\]

\[
S_{a+kn} = S_a + kS_n + \frac{k(k-1)nm}{2} + kma
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• **Additional conditions**

\[
\Delta_2(a, a) = \text{even}, \quad m > 0, \quad mn = \text{even}, \quad 2S_n = 0 \mod n.
\]

• **A mysterious condition** (the one we want but cannot prove):

\[
\Delta_3(a, b, c) = \text{even}
\]
• **n-cluster condition**: No off-particle zeros when \( c = n \) (or the wave function for the \( n \)-clusters is the Laughlin wave function or a \( n \)-electron cluster is “kind of trivial”)

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• **A mysterious condition** (the one we want but cannot prove):  

\[ \Delta_3(a, b, c) = \text{even} \]

• \((m; S_2, \cdots, S_n)\) that satisfy the above conditions correspond to *good* symmetric polynomials.  

\[ \rightarrow \] Those \((m; S_2, \cdots, S_n)\) “classify” symmetric polynomials and FQH states (with \( \nu = \frac{n}{m} \)).
The conditions are semi-linear →
if \((m; S_2, \cdots, S_n)\) and \((m'; S'_2, \cdots, S'_n)\) are solutions, then
\((m''; S''_2, \cdots, S''_n) = (m; S_2, \cdots, S_n) + (m'; S'_2, \cdots, S'_n)\) is also a
solution \(\sim P''(\{z_i\}) = P(\{z_i\})P'(\{z_i\})\)
Primitive solutions for pattern of zeros

The conditions are semi-linear $\rightarrow$

if $(m; S_2, \cdots, S_n)$ and $(m'; S_2', \cdots, S_n')$ are solutions, then

$(m''; S_2'', \cdots, S_n'') = (m; S_2, \cdots, S_n) + (m'; S_2', \cdots, S_n')$ is also a solution $\sim P''(\{z_i\}) = P(\{z_i\})P'(\{z_i\})$

1-cluster state: $\nu = 1/m$ Laughlin state

\[ P_{1/m} : \quad S = (m;), \]
\[ (n_0, \cdots, n_{m-1}) = (1, 0, \cdots, 0). \]

2-cluster state: Pfaffian state ($Z_2$ parafermion state)

\[ P_{2;Z_2} : \quad (m; S_2) = (2; 0), \]
\[ (n_0, \cdots, n_{m-1}) = (2, 0) \]

3-cluster state: $Z_3$ parafermion state

\[ P_{3;Z_3} : \quad (m; S_2, S_3) = (2; 0, 0), \]
\[ (n_0, \cdots, n_{m-1}) = (3, 0) \]
4-cluster state: $Z_4$ parafermion state

$$\begin{align*}
P_{\frac{4}{2};Z_4} : (m; S_2, \cdots, S_n) &= (2; 0, 0, 0), \\
(n_0, \cdots, n_{m-1}) &= (4, 0),
\end{align*}$$

5-cluster states: $Z_5$ (generalized) parafermion state

$$\begin{align*}
P_{\frac{5}{2};Z_5} : (m; S_2, \cdots, S_n) &= (2; 0, 0, 0, 0), \\
(n_0, \cdots, n_{m-1}) &= (5, 0)
\end{align*}$$

$$\begin{align*}
P_{\frac{5}{8};Z_5^{(2)}} : (m; S_2, \cdots, S_n) &= (8; 0, 2, 6, 10), \\
(n_0, \cdots, n_{m-1}) &= (2, 0, 1, 0, 2, 0, 0, 0)
\end{align*}$$

6-cluster state:

$$\begin{align*}
P_{\frac{6}{2};Z_6} : (m; S_2, \cdots, S_n) &= (2; 0, 0, 0, 0, 0), \\
(n_0, \cdots, n_{m-1}) &= (6, 0)
\end{align*}$$
7-cluster states:

\[ P_{7/2; Z_7} : (m; S_2, \cdots, S_n) = (2; 0, 0, 0, 0, 0, 0), \]
\[ (n_0, \cdots, n_{m-1}) = (7, 0) \]

\[ P_{7/8; Z_7}^{(2)} : (m; S_2, \cdots, S_n) = (8; 0, 0, 2, 6, 10, 14), \]
\[ (n_0, \cdots, n_{m-1}) = (3, 0, 1, 0, 3, 0, 0, 0) \]

\[ P_{7/18; Z_7}^{(3)} : (m; S_2, \cdots, S_n) = (18; 0, 4, 10, 18, 30, 42), \]
\[ (n_0, \cdots, n_{m-1}) = (2, 0, 0, 0, 0, 1, 0, 0, 2, 0, 0, 0, 0, 0) \]

\[ P_{7/14; Z_7} : (m; S_2, \cdots, S_n) = (14; 0, 2, 6, 12, 20, 28), \]
\[ (n_0, \cdots, n_{m-1}) = (2, 0, 1, 0, 1, 0, 1, 0, 2, 0, 0, 0, 0, 0, 0) \]

- Also get composite parafermion state \( P = P_{Z_{n_1}} P_{Z_{n_2}} \).
How good is the pattern-of-zero classification

Not so bad :-) Not so good :-(

- Every symm. poly. $P$ corresponds to a unique pattern of zero $\{S_a\}$. Only some patterns of zero correspond to a unique symm. poly.
- It appears that all the primitive patterns of zero correspond to a unique symm. poly.
- It is known that some composite patterns of zero $\rightarrow$ a unique symm. polynomial some composite patterns of zero $\rightarrow$ several symm. polynomials

So in general, we need more information than $\{n; m; S_a\}$ to fully characterize symmetry polynomial of infinite variables.
For those patterns of zeros that uniquely characterize a FQH wave function, we should be able to calculate the topological properties of FQH states from the data \((n; m; S_2, \cdots, S_n)\).
For those patterns of zeros that uniquely characterize a FQH wave function, we should be able to calculate the topological properties of FQH states from the data \((n; m; S_2, \ldots, S_n)\).

Physical properties that we want to get
- The filling fraction (actually given by \(\nu = n/m\)).
- Topological degeneracy on torus (and other Riemann surface)
- Number of quasiparticle types
- Quasiparticle charges
- Quasiparticle scaling dimensions
- Quasiparticle fusion algebra
- Quasiparticle statistics (Abelian and non-Abelian)
- The counting of edge excitations (central charge \(c\) and spectrum)
From Pattern of zeros to physical properties

- What are the physical properties of those FQH states described by $P_{1/2}$, $P_{1/4}$, and $P_{Pf}$? Are they really belong to different phases? What are the fractional charge/statistics of quasiparticles, edge excitations, etc?

- The densities of gapped FQH states are quantized as rational-number (filling fraction) $\nu \times \frac{1}{2\pi}$:

$$
\rho_e(z) = \frac{\int d^2z_2...d^2z_N |P(z, z_2, ..., z_N)|^2 e^{-\frac{1}{2} \sum |z_i|^2}}{\int d^2z_1d^2z_2...d^2z_N |P(z_1, z_2, ..., z_N)|^2 e^{-\frac{1}{2} \sum |z_i|^2}} = \left|z|<r_N\right| \nu \frac{1}{2\pi}
$$

$$
P_1 = \prod(z_i - z_j) \rightarrow \nu = 1.
$$

$$
P_{1/2} = \prod(z_i - z_j)^2 \rightarrow \nu = 1/2,
$$

$$
P_{1/4} = \prod(z_i - z_j)^4 \rightarrow \nu = 1/4,
$$

$$
P_{Pf} = Pf\left(\frac{1}{z_i - z_j}\right) \prod(z_i - z_j) \rightarrow \nu = 1.
$$

$\nu$ is quantized as exact rational number in $N \rightarrow \infty$ limit.
Why $\nu = 1$ for state $\Psi_1 = \prod_{i<j}(z_i - z_j)e^{-\sum |z_i|^2/4}$

One-particle eigenstate (orbital) for $H_0 = -\sum (\partial_z - \frac{B}{4} z^*)(\partial_{z^*} + \frac{B}{4} z)$:

$z' e^{-\frac{1}{4}|z|^2} \rightarrow$ a ring-like wave function with a radius $r_l = \sqrt{2l}$ and angular momentum $l$.

The $\nu = 1$ many-fermion state is obtained by filling the orbitals:

$$\Psi = \prod_{i<j}(z_i - z_j)e^{-\frac{1}{4}\sum |z_i|^2} = A[(z_1)^0(z_2)^1...e^{-\frac{1}{4}\sum |z_i|^2}$$

$\ell$ electrons within radius $r_l \rightarrow$ one electron per $\pi r_l^2 / l = 2\pi$ area.

$\rightarrow \nu = 1$. 

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Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
A general calculation of filling fraction $\nu$

- Total angular momentum $S_N$ of a $N$-electron FQH state

$$S_N = \frac{1}{2} \nu^{-1} N^2 + \# N + \#$$

The above is the minimal total angular momentum of the zero-energy state for the ideal Hamiltonian.

- For state $\Psi_{1/m} = \prod_{i<j} (z_i - z_j)^m e^{-\sum |z_i|^2 / 4}$, we have $S_N = \frac{m}{2} N(N - 1)$.

- The angular momentum of added $N^{th}$ electron

$$l_N = S_N - S_{N-1} \approx \partial S_N / \partial N = \nu^{-1} N + \#$$

Every electron occupy $\nu^{-1}$ orbitals $\rightarrow$ the filling fraction is $\nu$

$$P_1 = \prod (z_i - z_j) \rightarrow \nu = 1.$$ $$P_{1/2} = \prod (z_i - z_j)^2 \rightarrow \nu = 1/2.$$ $$P_{1/4} = \prod (z_i - z_j)^4 \rightarrow \nu = 1/4.$$ $$P_{1/4} = \text{Pf}(\frac{1}{\nu}) \prod (z_i - z_j) \rightarrow \nu = 1.$$
Quasi-holes in the $\nu = 1/m$ Laughlin state and fractional charge

- A hole-like excitation = missing an electron, charge = 1

$$\prod_i (\xi - z_i)^m \prod_{i<j} (z_i - z_j)^m e^{-\sum |z_i|^2/4}$$

which can be splitted into $m$ quasi-hole excitations:

$$\prod_i (\xi_1 - z_i) \cdots \prod_i (\xi_m - z_i) \prod_{i<j} (z_i - z_j)^m e^{-\sum |z_i|^2/4}$$

- A quasi-hole excitation = minimal excitation, charge = $1/m$

$$\prod_i (\xi - z_i) \prod_{i<j} (z_i - z_j)^m e^{-\sum |z_i|^2/4}$$

- Why the density dip have a small finite size?
Quasi-holes in the $\nu = 1/m$ Laughlin state and fractional statistics

**Calculate fractional statistics:**

- A hand-waving way

$$\text{Berry's phase} = 2\pi \times \text{encl. elec.}$$

- A quasiparticle is a bound state of $2\pi$ flux and $1/m$-charge.

$$\text{Berry's phase} = 2\pi \times (\text{encl. elec.} - 1/m)$$

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Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
Plasma approach to Laughline states

Why $\nu = 1/m$ for state $\Psi_1 = \prod_{i<j}(z_i - z_j)^m e^{-\sum |z_i|^2/4}$

- Consider the joint probability distribution of electron positions:

$$p(z_1 \cdots z_N) \propto |\Psi_1/m(z_1 \cdots z_N)|^2$$

$$= e^{-2m \sum_{i<j} \ln|z_i - z_j| - \frac{m}{2} \sum_i |z_i|^2} = e^{-\beta V(z_1 \cdots z_N)}$$

- Choosing $T = \frac{1}{\beta} = \frac{m}{2}$, we can view

$$V = -m^2 \sum_{i<j} \ln|z_i - z_j| + \frac{m}{4} \sum_i |z_i|^2$$

as the potential for a two-dimensional plasma of ‘charge’ $m$ particles.

- Each ‘charge’ $m$ particle in the plasma sees a potential

$$\phi(z) = |z|^2/4$$

which can be viewed as the potential produced by a uniform background ‘charge’ distribution with charge density

$$\rho_{\phi} = -1/2\pi \rightarrow \text{The ‘electric field’ } E(z) = \pi |z|^2 \rho_{\phi}/|z| = -|z|/2$$

- The plasma must be ‘charge’ neutral: $m \rho_e + \rho_{\phi} = 0 \rightarrow \rho_e = \frac{1}{m} \frac{1}{2\pi}$. 

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A ‘plasma’ calculation: $p(\{z_i\}) \propto e^{-\beta V}$:

$$V = -m^2 \sum_{i<j} \ln |z_i - z_j| - m \sum_i \ln |z_i - \xi| + \frac{m}{4} \sum_i |z_i|^2$$

Background charge density: $\rho_\phi = -\frac{1}{2\pi} + \delta(\xi)$
Fractional statistics = Berry phase of exch. quasi-holes

- A quasi-hole excitation = an \( N \)-electron state parameterized by \( \xi \):

\[
|\Psi_\xi\rangle = [N(\xi, \xi^*)]^{-1/2} \prod_i (\xi - z_i) \prod_{i<j} (z_i - z_j)^2 e^{-\sum |z_i|^2/4}
\]

- Berry's phase = phase change \( \Delta \varphi = a_\xi d\xi + a_{\xi^*} d\xi^* \)

\[
e^{-i\Delta \varphi} = \langle \Psi_\xi | \Psi_{\xi + d\xi} \rangle, \quad a_\xi = i \langle \Psi_\xi | \frac{\partial}{\partial \xi} | \Psi_\xi \rangle, \quad a_{\xi^*} = i \langle \Psi_\xi | \frac{\partial}{\partial \xi^*} | \Psi_\xi \rangle,
\]

- For unnormalized state \( \prod_i (\xi - z_i) \prod_{i<j} (z_i - z_j)^2 e^{-\sum |z_i|^2/4} \) that depends only on \( \xi \) (holomorphic), the Berry connection \( (a_\xi, a_{\xi^*}) \) can be calculated from the normalization \( N(\xi, \xi^*) \) of the holomorphic state:

\[
a_\xi = \frac{i}{2} \frac{\partial}{\partial \xi} \ln[N(\xi, \xi^*)], \quad a_{\xi^*} = -\frac{i}{2} \frac{\partial}{\partial \xi^*} \ln[N(\xi, \xi^*)].
\]
What is the Berry connection \((a_\xi, a_{\xi^*})\)?

We find \(N(\xi, \xi^*) = e^{\frac{1}{2m}|\xi|^2} \times \text{Const.}, \) since \(|\Psi_\xi|^2 = e^{-\beta V} \)

\[
V = -m^2 \sum_{i<j} \ln |z_i - z_j| - m \sum_i \ln |z_i - \xi| + \frac{1}{4} |\xi|^2 + \frac{m}{4} \sum_i |z_i|^2
\]

which is the total energy of a plasma of \(N \) ‘charge’-\(m\) particles at \(z_i\) and one ‘charge’-1 particle at \(\xi\).

Due to the screening, the average \(\langle V \rangle\) does not depend on \(\xi \rightarrow \langle \Psi_\xi | \Psi_\xi \rangle\) does not depend on \(\xi\) and is normalized.

We find \(a_\xi = i \frac{1}{4m} \xi^*, \ a_{\xi^*} = -i \frac{1}{4m} \xi\) which describes a uniform ‘magnetic’ field.

Berry’s phase for a loop:

\[
\oint_C (a_\xi d\xi + a_{\xi^*} d\xi^*) = 2\pi \frac{\text{Area encl. by } C}{2\pi m} = 2\pi \times \text{num. of encl. elec.}
\]

which one can see directly from the wave function

\[
\prod_i (\xi - z_i) \prod_{i<j} (z_i - z_j)^2 e^{-\sum |z_i|^2/4}.
\]
Calculate fractional statistics

The Berry connection for two quasi-holes: The wave function

\[ \Psi_{\xi,\xi'} = N^{-1/2} \prod_i (\xi - z_i) \prod_i (\xi' - z_i) \prod_{i<j} (z_i - z_j)^2 e^{-\sum |z_i|^2/4} \]

\[ N(\xi, \xi^*, \xi', \xi'^*) = e^{\frac{1}{2m}(|\xi|^2 + |\xi'|^2) + \frac{1}{m} \ln |\xi - \xi'|^2} \times \text{Const.} \]

\[ V = -m \sum_i \left[ \ln |z_i - \xi| + \ln |z_i - \xi'| \right] + \frac{1}{4} [ |\xi|^2 + |\xi'|^2 ] - \ln |\xi - \xi'| \]

\[ - m^2 \sum_{i<j} \ln |z_i - z_j| + \frac{m}{4} \sum_i |z_i|^2 \]

\[ a_\xi = i \frac{1}{4m} \xi^* - \frac{i}{2m} \frac{1}{\xi - \xi'}, \quad a_{\xi^*} = -i \frac{1}{4m} \xi + \frac{i}{2m} \frac{1}{\xi^* - \xi'^*} \]
Quasi-holes in the $\nu = 1$ Pfaffian state: What is non-Abelian statistics?

- Ground state: $z_1 \approx z_2$, no zero; $z_1 \approx z_2 \approx z_3$, second-order zero;

$$\Psi_{Pf} = \mathcal{A}\left(\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots \frac{1}{z_{N-1} - z_N}\right) = \text{Pf}\left(\frac{1}{z_i - z_j}\right)$$

- A charge-1 quasi-hole state

$$\Psi_{\text{charge-1}} = \prod (\xi - z_i) \mathcal{A}\left(\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots \frac{1}{z_{N-1} - z_N}\right)$$

$$= \mathcal{A}\left(\frac{(\xi - z_1)(\xi - z_2)(\xi - z_3)(\xi - z_4)}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots \right) = \text{Pf}\left(\frac{(\xi - z_i)(\xi - z_j)}{z_i - z_j}\right)$$

- A state with two charge-1/2 quasi-holes

$$\Psi_{(\xi)(\xi')} = \mathcal{A}\left(\frac{(\xi - z_1)(\xi' - z_2) + (1 \leftrightarrow 2)}{z_1 - z_2} \frac{(\xi - z_3)(\xi' - z_4) + (3 \leftrightarrow 4)}{z_3 - z_4} \cdots \right)$$

$$= \text{Pf}\left(\frac{(\xi - z_i)(\xi' - z_j) + (\xi - z_j)(\xi' - z_i)}{z_i - z_j}\right)$$
How many states with four charge-1/2 quasi-holes?

- One of the state with four charge-1/2 quasi-holes

\[
P_{(12)(34)} = \text{Pf}\left(\frac{(\xi_1 - z_i)(\xi_2 - z_i)(\xi_3 - z_j)(\xi_4 - z_j) + (i \leftrightarrow j)}{z_i - z_j}\right) \ldots
\]

\[
= \text{Pf}\left(\frac{[12, 34]_{z_i z_j}}{z_i - z_j}\right) \ldots
\]

The other two are \( P_{(13)(14)}, P_{(14)(23)} \).
How many states with four charge-1/2 quasi-holes?

- One of the state with four charge-1/2 quasi-holes

\[
P_{(12)(34)} = \text{Pf}\left( \frac{(\xi_1 - z_i)(\xi_2 - z_i)(\xi_3 - z_j)(\xi_4 - z_j) + (i \leftrightarrow j)}{z_i - z_j} \right) \ldots
\]

\[
= \text{Pf}\left( \frac{[12, 34]_z z_j}{z_i - z_j} \right) \ldots
\]

The other two are \( P_{(13)(14)}, P_{(14)(23)}. \)

- **But only two linearly independent states.** Using Nayak & Wilczek

\[
[12, 34]_z z_j - [13, 24]_z z_j = (z_i - z_j)^2(\xi_1 - \xi_4)(\xi_2 - \xi_3) = z_{ij}^2 \xi_1 \xi_4 \xi_2 \xi_3
\]

we find (with \( z_{12} = z_1 - z_2, \xi_{12} = \xi_1 - \xi_2, \) etc)

\[
P_{(13)(24)} = A\left( \frac{[12, 34]_z z_2 - z_{12}^2 \xi_1 \xi_2 \xi_3}{z_12} \frac{[12, 34]_z z_4 - z_{34}^2 \xi_1 \xi_3 \xi_2}{z_34} \ldots \right)
\]

\[
= P_{(12)(34)} - N_{\text{pair}} A\left( z_{12} \xi_1 \xi_2 \xi_3 \frac{[12, 34]_z z_2}{z_{34}} \ldots \right)
\]
\[ P_{(12)(34)} - P_{(13)(24)} = N_{pair} \xi_{14} \xi_{23} A(z_{12} \frac{[12, 34]z_3 z_4}{z_{34}} \ldots) \]

Similarly

\[ P_{(12)(34)} - P_{(14)(23)} = N_{pair} \xi_{13} \xi_{24} A(z_{12} \frac{[12, 34]z_3 z_4}{z_{34}} \ldots) \]

So

\[ \frac{P_{(12)(34)} - P_{(13)(24)}}{\xi_{14} \xi_{23}} = \frac{P_{(12)(34)} - P_{(14)(23)}}{\xi_{13} \xi_{24}} \]

- Two states for four charge-1/2 quasiholes, even if we fixed their positions. The two states are topologically degenerate.

\[ D_n = \frac{1}{2} (\sqrt{2})^n \text{ states for } n \text{ charge-1/2 quasiholes.} \]

\[ \sqrt{2} \text{ states per charge-1/2 quasihole} \]

Quantum dimension for the charge-1/2 quasihole \( d = \sqrt{2} \)

The charge-1/2 quasihole has a non-Abelian statistics. For Abelian anyons \( d = 1 \).

- In general, for \( n \) particles with non-Abelian statistics, there are \( D_n \) degenerate states even after we fix the positions of the particle:

\[ |\alpha, \xi_1, \ldots, \xi_n\rangle, \quad \alpha = 1, \ldots, D_n. \]
Local and topological quasiparticle excitations

In a system: \( H = \sum_x H_x \)

- a particle-like excitation:
  
  energy density \( = \langle \Psi_{exc}|H_x|\Psi_{exc}\rangle \)

- Local quasiparticle excitation: \( |\Psi_{exc}\rangle = \hat{O}(x)|\Psi_{grnd}\rangle \)

- Topological quasiparticle excitations \( |\Psi_{exc}\rangle \neq \hat{O}(x)|\Psi_{grnd}\rangle \) for any local operators \( \hat{O}(x) \)

- Topological quasiparticle types:
  
  if \( |\Psi'_{exc}\rangle = \hat{O}(x)|\Psi_{exc}\rangle \), then \( |\Psi'_{exc}\rangle \) and \( |\Psi_{exc}\rangle \) belong to the same type.

\[ \text{Number of topological quasiparticle types is an important topological invariant that (partially) characterizes the topological order.} \]

An amazing relation:

\[ \text{Number of topological quasiparticle types} \]
\[ = \text{Ground states degeneracy on torus} \]
The topological types of quasiparticles in Laughlin state

- A quasiparticle is a defect in the ground state wave function $P(\{z_i\})$. It is a place where we have more power of zeros.
- For example: Ground state $\prod_{i<j}(z_i - z_j)^m$.
  A quasiparticle state $\prod_i(z_i - \xi)\prod_{i<j}(z_i - z_j)^m$
- Density distribution of a quasiparticle state:

$$Q$$

The defect at $\eta$ is point-like and carries a charge $Q$.

- A more general quasiparticle state $\prod_i(z_i - \xi)^k\prod_{i<j}(z_i - z_j)^m$
- There are $m$ types of quasiparticles labeled by $k = 0, 1, \cdots, m-1$ for the $\nu = 1/m$ Laughlin state.
  The quasiparticles labeled by $k$ and $k + m$ differ by one electron and are regarded as the same type.
- There are $m$ degenerate ground states on torus for the $\nu = 1/m$ Laughlin state.

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Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
A quasiparticle $\gamma$ in a FQH state can also be quantitatively characterized by pattern of zeros $\{S_{\gamma;a}\}$:

- $P_\gamma(\eta; \{z_i\})$ has a quasiparticle at $z = \eta$
- Let $z_i = \lambda \xi_i + \eta$, $i = 1, 2, \cdots, a$ (bring $a$ electrons to the quasiparticle)

\[
P_\gamma(\eta; \{z_i\}) = \lambda^{S_{\gamma;a}} P_\gamma(z^{(a)} = \eta, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_{\gamma;a}+1})
\]

$S_{\gamma;a}$ is the order of zero of $\Psi_\gamma(\xi, z_i)$ when we bring $a$ electrons to $\xi$. 

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The pattern of zeros for quasiparticles

- A quasiparticle $\gamma$ in a FQH state can also be quantitatively characterized by pattern of zeros $\{S_{\gamma;a}\}$:
  - $P_\gamma(\eta; \{z_i\})$ has a quasiparticle at $z = \eta$
  - Let $z_i = \lambda \xi_i + \eta$, $i = 1, 2, \cdots, a$ (bring $a$ electrons to the quasiparticle)

$$P_\gamma(\eta; \{z_i\}) = \lambda^{S_{\gamma;a}} P_\gamma(z^{(a)} = \eta, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_{a+1}})$$

$S_{\gamma;a}$ is the order of zero of $\Psi_\gamma(\xi, z_i)$ when we bring $a$ electrons to $\xi$.

The sequence of integers $\{S_{\gamma;a}\}$ characterizes the quasiparticle $\gamma$.

- $\{S_a\}$ correspond to the trivial quasiparticle $\gamma = 0$: $\{S_{0;a}\} = \{S_a\}$
A quasiparticle $\gamma$ in a FQH state can also be quantitatively characterized by pattern of zeros $\{S_{\gamma;a}\}$:

- $P_\gamma(\eta; \{z_i\})$ has a quasiparticle at $z = \eta$
- Let $z_i = \lambda \xi_i + \eta$, $i = 1, 2, \cdots, a$ (bring $a$ electrons to the quasiparticle)

$$P_\gamma(\eta; \{z_i\}) = \lambda^{S_{\gamma;a}} P_\gamma(z^{(a)} = \eta, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_{a+1}})$$

$S_{\gamma;a}$ is the order of zero of $\Psi_\gamma(\xi, z_i)$ when we bring $a$ electrons to $\xi$. The sequence of integers $\{S_{\gamma;a}\}$ characterizes the quasiparticle $\gamma$.

- $\{S_a\}$ correspond to the trivial quasiparticle $\gamma = 0$: $\{S_{0;a}\} = \{S_a\}$
- To find the allowed quasiparticles, we simply need to find (i) the conditions that $S_{\gamma;a}$ must satisfy and (ii) all the $S_{\gamma;a}$ that satisfy those conditions.
Conditions on $S_{\gamma; a}$

- Concave condition

$$S_{\gamma; a+b} - S_{\gamma; a} - S_b \geq 0,$$
$$S_{\gamma; a+b+c} - S_{\gamma; a+b} - S_{\gamma; a+c} - S_{b+c} + S_{\gamma; a} + S_b + S_c \geq 0$$

- Taking $b = c = 1$, we obtain

$$S_{\gamma; a+2} - 2S_{\gamma; a+1} + S_{\gamma; a} = l_{\gamma; a+2} - l_{\gamma; a+1} \geq S_2, \quad a \geq 0.$$

$l_{\gamma; a}$ increases with $a$: $l_{\gamma; a+1} - l_{\gamma; a} \geq S_2 \geq 0$.

$z^{(a)}$ is a quasiparticle $\gamma$ fused with $a$ electrons.
• \( n \)-cluster condition

\[
S_{\gamma};a+kn = S_{\gamma};a + k(S_{\gamma};n + ma) + mn\frac{k(k-1)}{2}
\]

\((S_{\gamma};1, \cdots, S_{\gamma};n)\) determine all \( \{S_{\gamma};a\} \).

• From the above, we can show that

\[
l_{\gamma};a+n = S_{\gamma};a+n - S_{\gamma};a-1+n = l_{\gamma};a + m \rightarrow n_{l+m} = n_{l}.
\]

The occupation pattern \( n_{\gamma;l} \) is periodic:

Unit cell = \( m \) orbitals occupied by \( n \) electrons
• $n$-cluster condition

\[ S_{\gamma;a+kn} = S_{\gamma;a} + k(S_{\gamma;n} + ma) + mn \frac{k(k - 1)}{2} \]

$(S_{\gamma;1}, \cdots, S_{\gamma;n})$ determine all $\{S_{\gamma;a}\}$.

• From the above, we can show that

\[ l_{\gamma;a+n} = S_{\gamma;a+n} - S_{\gamma;a-1+n} = l_{\gamma;a} + m \rightarrow n_{l+m} = n_{l}. \]

The occupation pattern $n_{\gamma;l}$ is periodic:

Unit cell = $m$ orbitals occupied by $n$ electrons

• Find all $(S_{\gamma;1}, \cdots, S_{\gamma;n})$ that satisfy that above conditions

\[ \rightarrow \text{obtain all the quasiparticles}. \]
For the $\nu = 1$ Pfaffian state ($n = 2$ and $m = 2$)

$$S_1, S_2, \cdots : 0, 0, 2, 4, 8, 12, 18, 24, \cdots$$

$$n_0 n_1 n_2 \cdots : 202020202020202020202\cdots$$

- Quasiparticle solutions ($S_{\gamma;a} \rightarrow l_{\gamma;a} \rightarrow n_{\gamma;l}$):

  $$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 202020202020202020202\cdots \quad Q_{\gamma} = 0$$

  $$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 020202020202020202020\cdots \quad Q_{\gamma} = 1$$

  $$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 111111111111111111111\cdots \quad Q_{\gamma} = 1/2$$

- All other quasiparticle solutions can be obtained from the above three by removing extra electrons $\rightarrow$ only 3 quasiparticle types.
For the $\nu = 1$ Pfaffian state ($n = 2$ and $m = 2$)

$$S_1, S_2, \cdots : 0, 0, 2, 4, 8, 12, 18, 24, \cdots$$

$$n_0 n_1 n_2 \cdots : 202020202020202020202\cdots$$

- Quasiparticle solutions ($S_{\gamma;a} \rightarrow l_{\gamma;a} \rightarrow n_{\gamma;l}$):

  $$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 202020202020202020202\cdots \quad Q_{\gamma} = 0$$

  $$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 020202020202020202020\cdots \quad Q_{\gamma} = 1$$

  $$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 1111111111111111111\cdots \quad Q_{\gamma} = 1/2$$

- All other quasiparticle solutions can be obtained from the above three by removing extra electrons → only 3 quasiparticle types.

- Ground state degeneracy on torus = number of quasiparticle types

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For the $\nu = 1$ Pfaffian state ($n = 2$ and $m = 2$)

$$S_1, S_2, \cdots : 0, 0, 2, 4, 8, 12, 18, 24, \cdots$$

$$n_0 n_1 n_2 \cdots : 20202020202020202020 \cdots$$

- Quasiparticle solutions ($S_\gamma; a \to l_\gamma; a \to n_\gamma; l$):

  $$n_\gamma; 0 n_\gamma; 1 n_\gamma; 2 \cdots : 20202020202020202020202020 \cdots \quad Q_\gamma = 0$$

  $$n_\gamma; 0 n_\gamma; 1 n_\gamma; 2 \cdots : 020202020202020202020202020 \cdots \quad Q_\gamma = 1$$

  $$n_\gamma; 0 n_\gamma; 1 n_\gamma; 2 \cdots : 1111111111111111111111 \cdots \quad Q_\gamma = 1/2$$

- All other quasiparticle solutions can be obtained from the above three by removing extra electrons $\to$ only 3 quasiparticle types.

- Ground state degeneracy on torus $=$ number of quasiparticle types

- Charge of quasiparticles

$$Q_\gamma = \frac{1}{m} \sum_{a=1}^{n} (l_\gamma; a - l_a)$$

the average increase of $l_\gamma; a$ from the ground state values $l_a$. 

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Pattern of zeros and generalized Pauli exclusion rule

- \( l_{\gamma;a+1} - l_{\gamma;a} \geq S_2 \rightarrow \) the spacing between any two occupied orbitals by different electrons is no less than \( S_2 \)
  
  \( \rightarrow \) a generalized Pauli exclusion rule

**More general Pauli exclusion (GPE) rule**

In terms of \( l_{\gamma;a} = S_{\gamma;a} - S_{\gamma;a-1} \), the concave condition for quasiparticles becomes

\[
\sum_{k=1}^{b} l_{\gamma;a+k} \geq S_b = \sum_{k=1}^{b} l_{a+k}, \quad \rightarrow \quad l_{\gamma;a} \geq l_a,
\]

\[
\sum_{k=1}^{c} (l_{\gamma;a+b+k} - l_{\gamma;a+k}) \geq S_{b+c} - S_b - S_c = \sum_{k=1}^{c} (l_{b+k} - l_k)
\]

for any \( a, b, c \in \mathbb{Z}_+ \). Note that \( l_{\gamma;a+b} - l_{\gamma;a} \) is the spread of \( b+1 \) electrons. Setting \( c = 1 \rightarrow \) The spread of \( b \) electrons must be \( l_b \) or more. (\( l_b = \) the spread of the first \( b \) electrons in the ground state.)
For the $\nu = 1$ Pfaffian state ($n = 2$ and $m = 2$)

$$S_1, S_2, \cdots : 0, 0, 2, 4, 8, 12, 18, 24, \cdots$$

$$l_1 l_2 l_3 \cdots : 0, 0, 2, 2, 4, 4, 6, 6, 8, 8, \cdots$$

$$n_0 n_1 n_2 \cdots : 2020202020202020202 \cdots ,$$

- **GPE rule:**  
  1. The spread of three particles $\geq l_3 - l_1 = 2$.  
  2. The spread of two particles plus the spread of next two particles $\geq (l_2 - l_1) + (l_3 - l_2) = 2$. (3) ...

- the quasiparticle occupation patterns

  $$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 2020202020202020202 \cdots \quad Q_{\gamma} = 0$$

  $$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 0202020202020202020 \cdots \quad Q_{\gamma} = 1$$

  $$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 111111111111111111111 \cdots \quad Q_{\gamma} = 1/2$$

are the only **close-pacted** occupation patterns satisfying the above generalized Pauli exclusion rules.

- **close-pacted:**
  1. add any electrons $\rightarrow$ violate the GPE rules.
Topological degeneracy

- Symmetric polynomial $\rightarrow$ FQH state on plane (sphere).
  Holomorphic function with quasiperiodic bound condition in $x$- and $y$-directions $\rightarrow$ FQH state on torus.
- How many holomorphic functions with a pattern of zero $(n; m; S_a)$ can be put on a torus?
  How many zero-energy states the ideal FQH Hamiltonian have on a torus?
- The ground state degeneracy is split for non-ideal Hamiltonian, but is recovered in $N \rightarrow \infty$ limit even for non-ideal Hamiltonian!

So the ground state degeneracy in $N \rightarrow \infty$ limit is a new quantum number that characterizes different states of matter.
New quantum number $\rightarrow$ new states of matter with non-trivial topological order.  Wen 89
Consider a quasiparticle $\gamma$ on the disk at $z = 0$, whose type is determined by $n_{\gamma,l}$ in large $l$ limit.

- On the disk, the wave function has the POZ $S_a$ everywhere with $z \neq 0 \rightarrow$ zero energy.
- Deform the disk into a torus
- On the torus, the wave function has the POZ $S_a$ everywhere $\rightarrow$ zero energy.
- The number of the zero-energy patterns (wave functions) is the same as the numbers of quasiparticle types.
Number of quasiparticle types from pattern of zeros

For the parafermion states $P_{\frac{n}{2}}; z_n \ (m = 2)$,

<table>
<thead>
<tr>
<th>$P_{\frac{2}{2}}; z_2$</th>
<th>$P_{\frac{3}{2}}; z_3$</th>
<th>$P_{\frac{4}{2}}; z_4$</th>
<th>$P_{\frac{5}{2}}; z_5$</th>
<th>$P_{\frac{6}{2}}; z_6$</th>
<th>$P_{\frac{7}{2}}; z_7$</th>
<th>$P_{\frac{8}{2}}; z_8$</th>
<th>$P_{\frac{9}{2}}; z_9$</th>
<th>$P_{\frac{10}{2}}; z_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

For the parafermion states $P_{\frac{n}{2 + 2n}}; z_n \ (m = 2 + 2n)$

<table>
<thead>
<tr>
<th>$P_{\frac{2}{6}}; z_2$</th>
<th>$P_{\frac{3}{8}}; z_3$</th>
<th>$P_{\frac{4}{10}}; z_4$</th>
<th>$P_{\frac{5}{12}}; z_5$</th>
<th>$P_{\frac{6}{14}}; z_6$</th>
<th>$P_{\frac{7}{16}}; z_7$</th>
<th>$P_{\frac{8}{18}}; z_8$</th>
<th>$P_{\frac{9}{20}}; z_9$</th>
<th>$P_{\frac{10}{22}}; z_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>16</td>
<td>25</td>
<td>36</td>
<td>49</td>
<td>64</td>
<td>81</td>
<td>100</td>
<td>121</td>
</tr>
</tbody>
</table>

For the generalized parafermion states $P_{\nu} = \frac{n}{m}; z_n^{(k)}$

<table>
<thead>
<tr>
<th>$P_{\frac{5}{8}}; z_5^{(2)}$</th>
<th>$P_{\frac{5}{18}}; z_5^{(2)}$</th>
<th>$P_{\frac{7}{8}}; z_7^{(2)}$</th>
<th>$P_{\frac{7}{22}}; z_7^{(2)}$</th>
<th>$P_{\frac{7}{18}}; z_7^{(3)}$</th>
<th>$P_{\frac{7}{32}}; z_7^{(3)}$</th>
<th>$P_{\frac{8}{18}}; z_8^{(3)}$</th>
<th>$P_{\frac{9}{8}}; z_9^{(2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>54</td>
<td>32</td>
<td>88</td>
<td>72</td>
<td>128</td>
<td>81</td>
<td>40</td>
</tr>
</tbody>
</table>

where $k$ and $n$ are coprime.
For the composite parafermion states $P_{\frac{n_1}{m_1} ; z_{n_1}^{(k_2)}} P_{\frac{n_2}{m_2} ; z_{n_2}^{(k_2)}}$ obtained as products of two parafermion wave functions

<table>
<thead>
<tr>
<th>$P_{\frac{2}{2} ; z_2} P_{\frac{3}{2} ; z_3}$</th>
<th>$P_{\frac{3}{2} ; z_3} P_{\frac{4}{2} ; z_4}$</th>
<th>$P_{\frac{2}{2} ; z_2} P_{\frac{5}{2} ; z_5}$</th>
<th>$P_{\frac{2}{2} ; z_2} P_{\frac{5}{8} ; z_5^{(2)}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>70</td>
<td>63</td>
<td>117</td>
</tr>
</tbody>
</table>

where $n_1$ and $n_2$ are coprime. The inverse filling fractions of the above composite states are $\frac{1}{\nu} = \frac{1}{\nu_1} + \frac{1}{\nu_2} = \frac{m_1}{n_1} + \frac{m_2}{n_2}$.

- Those results from the pattern of zeros all agree with the results from parafermion CFT: Barkeshli & Wen, 2008

$$\# \text{ of quasiparticles} = \frac{1}{\nu} \prod_i \frac{n_i(n_i + 1)}{2}$$

for the generalized composite parafermions state

$$P = \prod_i P_{\frac{n_i}{m_i} ; z_{n_i}^{(k_i)}}, \quad \{n_i\} \text{ coprime, } \ (k_i, n_i) \text{ coprime.}$$

$$\frac{1}{\nu} = \sum m_i/n_i$$
Edge excitations and conformal field theory

- Incompressible liquid on compact space (sphere) ⇒ finite energy gap, and no low energy excitations.
- A droplet of incompressible liquid can change its shape ⇒ gapless edge excitations.

Dispersion of edge excitations:
- Ground state \( \prod (z_i - z_j)^m e^{-|z_i|^2/4} \) – circular droplet

- Edge excitations – deformed droplet
  Deformed droplet is formed by moving electron away from the center
  ⇒ Edge excitations always have larger angular momentum than ground state
  ⇒ Edge excitations always move in one direction along the edge
\( \nu = 1/2 \) Laughlin state:

- for ideal Hamiltonian \( V_{1/2}(z_1, z_2) = \delta(z_1 - z_2) \), the \( N \) electron state \( P_{1/2} = \prod_{i<j}(z_i - z_j)^2 \), is the zero-energy state with minimal total angular momentum (the order of \( z_i \)'s) \( M_0 = N(N-1) \).
- Other zero-energy state is obtained by deforming the Laughlin wave function without reducing the order of zeros.
- \( \psi_{\text{edge}} = P_{\text{sym}}(\{z_i\})\psi_{1/2} \) where \( P_{\text{sym}} \) is a symmetric polynomial, such as
  \[ \sum z_i, \ (\sum z_i)^2, \ \sum z_i^2, \ldots \]

<table>
<thead>
<tr>
<th>( P_{\text{sym}} )</th>
<th>( M_0 )</th>
<th>( M_0 + 1 )</th>
<th>( M_0 + 2 )</th>
<th>( M_0 + 3 )</th>
<th>( M_0 + 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td># of states</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>( \sum z_i )</td>
<td>1</td>
<td>( \sum z_i )</td>
<td>( (\sum z_i)^2 )</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>( \sum z_i^2 )</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
</tbody>
</table>
Edge excitations of ideal Hamiltonian

- \( \nu = 1 \) Pfaffian state: For ideal Hamiltonian

\[
S[\nu_0 \delta(z_1 - z_2)\delta(z_2 - z_3) - \nu_1 \delta(z_1 - z_2)\partial_{z_3^*}\delta(z_2 - z_3)\partial z_3],
\]

\( \Psi_{Pfa} = A\left(\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots \right) \prod_{i<j}(z_i - z_j) \), is the zero-energy state with minimal total angular momentum.

- Other zero-energy states are given by \( \Psi_{edge} = P_{sym}(\{z_i\})\Psi_{Pfa} \) or more generally

\[
\Psi_{edge} = A \left( P_{any}(\{z_i\}) \frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots \right) \prod_{i<j}(z_i - z_j),
\]

where \( P_{any} \) is any polynomial.

- The counting is very difficult. Independent \( P_{any} \) may generate linearly dependent wave function.

<table>
<thead>
<tr>
<th>( M )</th>
<th>( M_0 )</th>
<th>( M_0 + 1 )</th>
<th>( M_0 + 2 )</th>
<th>( M_0 + 3 )</th>
<th>( M_0 + 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td># of states</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

What kind of theory produce the above spectrum?
Low energy effective theory of edge excitations

- The $\nu = 1/m$ state $\Psi_{1/m} = \prod (z_i - z_j)^m$.  
  $\Rightarrow$ edge excitations = edge waves 
  $\Rightarrow$ edge phonons after quantization. 
  Displacement $h \propto$ 1D edge density $\rho = h\rho_{2D}$.
- Confining electric field induces edge current

$$j = \sigma_{xy} \hat{z} \times E, \quad \sigma_{xy} = \nu \frac{e^2}{2\pi \hbar}$$

- Electron drift velocity at the edge

$$\nu = \frac{E}{B} c$$

- The wave equation for the propagating edge wave:

$$\partial_t \rho + \nu \partial_x \rho = 0, \quad \rho(x) = nh(x)$$

- The Hamiltonian (i.e. the energy) of the edge waves:

$$H = \int dx \frac{1}{e \rho} E \hbar = \int dx \frac{\nu}{\pi} \rho^2$$

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• In momentum space

\[ \dot{\rho}_k = -i \nu k \rho_k, \quad H = 2\pi \frac{V}{\nu} \sum_{k>0} \rho_{-k} \rho_k \]

• If we identify \( \rho_k|_{k>0} \) as the ‘coordinates’ and \( \pi_k = i2\pi \rho_{-k}/\nu k \) as the corresponding canonical ‘momenta’, then the standard Hamiltonian equation

\[ \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \]

will reproduce the equation of motion \( \dot{\rho}_k = i\nu k \rho_k \)

• The phase-space Lagrangian:

\[ L = \int dx \frac{\pi}{\nu} \rho \frac{\partial t}{\partial x} \rho - \int dx \frac{\pi V}{\nu} \rho^2 = \frac{m}{4\pi} \int dx \partial_x \phi (\partial_t - \nu \partial_x) \phi, \]

\[ \rho = \frac{1}{2\pi} \partial_x \phi \]
**Quantization**: view $\rho_k$ and $\pi_k$ as operators that satisfy $[\rho_k, \pi_{k'}] = i \delta_{kk'}$. After quantization we have

$$[\rho_k, \rho_{k'}] = \frac{\nu}{2\pi} k \delta_{k+k'}, \quad k, k' = \text{integer} \times \frac{2\pi}{L}$$

$$H = 2\pi \nu \sum_{k>0} \rho_{-k} \rho_k = \sum_{k>0} \nu k a_k^\dagger a_k$$

<table>
<thead>
<tr>
<th>$M$</th>
<th>$M_0$</th>
<th>$M_0 + 1$</th>
<th>$M_0 + 2$</th>
<th>$M_0 + 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td># of states $D_n$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>phonons</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$$D_n \approx \frac{1}{4\sqrt{3}n} e^{\pi \sqrt{c} \sqrt{\frac{2n}{3}}} \approx e^{\pi \sqrt{c} \sqrt{\frac{2n}{3}}}, \quad c = 1,$$

where $c$ is the *central charge*. 
Edge theory for the Pfaffian state

What edge theory generates the spectrum of the Pfaffian state?

<table>
<thead>
<tr>
<th>$M$</th>
<th>$D_n$</th>
<th>$M_0$</th>
<th>$M_0 + 1$</th>
<th>$M_0 + 2$</th>
<th>$M_0 + 3$</th>
<th>$M_0 + 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td># of states</td>
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<td>1</td>
<td>3</td>
<td>5</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

The above theory reproduces the edge spectrum (for neutral even-fermion excitations) found in numerical calculations:

The central charge is $c = 3/2$, since $D_n \approx e^{\pi \sqrt{c} \sqrt{2} n}$. 

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Edge theory for the Pfaffian state

What edge theory generates the spectrum of the Pfaffian state?

<table>
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<tr>
<th># of states</th>
<th>$D_n$</th>
<th>$M$</th>
<th>$M_0$</th>
<th>$M_0 + 1$</th>
<th>$M_0 + 2$</th>
<th>$M_0 + 3$</th>
<th>$M_0 + 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$M_0$</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

- Pfaffian edge state = edge phonon + something

$$H = \sum_{k>0, k \in \mathbb{Z}} vka_k^{\dagger}a_k + \sum_{k>0, k \in \mathbb{Z} + \frac{1}{2}} vk\psi_k^{\dagger}\psi_k$$

chiral phonon

chiral fermion (real)

The above theory reproduces the edge spectrum (for neutral even-fermion excitations) found in numerical calculations:

$$(1, 1, 2, 3, 5, 10...) \otimes (1, 0, 1, 1, 2, ...) = (1, 1, 3, 5, 10, ...)$$

How about fermions with periodic boundary condition with $k \in \mathbb{Z}$

- The central charge is $c = 3/2$, since $D_n \approx e^{\pi\sqrt{c}\sqrt{\frac{2n}{3}}}$, $c = 3/2$. 

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Find the edge spectrum (with even fermions) of the following edge Hamiltonian

\[ H = \sum_{k > 0, k \in \mathbb{Z}} v k a_k^\dagger a_k + \sum_{k \geq 0, k \in \mathbb{Z}} v k \psi_k^\dagger \psi_k \]

where fermions have a periodic boundary condition with \( k \in \mathbb{Z} \).
Using the many-body density of the states

\[ D_n \approx e^{\pi \sqrt{c} \sqrt{\frac{2n}{3}}} \]

to compute the heat capacity of the system. Note that \( D_n \) is the number many-body states with angular momentum \( M = M_0 + n\hbar \). Assume those states all have the same energy \( E_n = p_n v, \quad p_n = \frac{2\pi n\hbar}{L}, \) where \( v \) is the velocity of the edge excitations and \( L \) the length of the edge.

(Hint: the heat capacity has a form \( C = \frac{#cLT}{v}, \) where \( T \) is the temperature)
GPE rule and edge spectrum for the Pfaffian state

For the $\nu = 1$ Pfaffian state ($n = 2$ and $m = 2$)

$$S_1, S_2, \cdots: 0, 0, 2, 4, 8, 12, 18, 24, \cdots$$

$$l_1 l_2 l_3 \cdots: 0, 0, 2, 2, 4, 4, 6, 6, 8, 8, \cdots$$

$$n_0 n_1 n_2 \cdots: 202020202020202020202\cdots,$$

- **GPE rule:** *The spread of three particles* $\geq l_3 - l_1 = 2$.
- **edge occupation patterns**

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots: 20202020202|000000000\cdots \quad Q_{\gamma} = 0$$

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots: 0202020202|000000000\cdots \quad Q_{\gamma} = 1$$

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots: 1111111111|000000000\cdots \quad Q_{\gamma} = 1/2$$

$\rightarrow$ two kinds of edges.
First kind of edge:

\[
M_0 : n_\gamma;0 n_\gamma;1 n_\gamma;2 \cdots : 20202020202|00000000 \cdots \\
M_0 + 1 : n_\gamma;0 n_\gamma;1 n_\gamma;2 \cdots : 20202020201|10000000 \cdots \\
M_0 + 1 : n_\gamma;0 n_\gamma;1 n_\gamma;2 \cdots : 20202020112|00000000 \cdots \\
M_0 + 2 : n_\gamma;0 n_\gamma;1 n_\gamma;2 \cdots : 20202020201|01000000 \cdots \\
M_0 + 2 : n_\gamma;0 n_\gamma;1 n_\gamma;2 \cdots : 20202020200|20000000 \cdots \\
M_0 + 2 : n_\gamma;0 n_\gamma;1 n_\gamma;2 \cdots : 20202020111|10000000 \cdots \\
M_0 + 3 : n_\gamma;0 n_\gamma;1 n_\gamma;2 \cdots : 20202020201|00100000 \cdots \\
M_0 + 3 : n_\gamma;0 n_\gamma;1 n_\gamma;2 \cdots : 20202020200|11000000 \cdots \\
M_0 + 3 : n_\gamma;0 n_\gamma;1 n_\gamma;2 \cdots : 20202020111|01000000 \cdots \\
M_0 + 3 : n_\gamma;0 n_\gamma;1 n_\gamma;2 \cdots : 20202020110|20000000 \cdots \\
M_0 + 3 : n_\gamma;0 n_\gamma;1 n_\gamma;2 \cdots : 20202011111|10000000 \cdots \\
\]

<table>
<thead>
<tr>
<th># of states $D_n$</th>
<th>$M_0$</th>
<th>$M_0 + 1$</th>
<th>$M_0 + 2$</th>
<th>$M_0 + 3$</th>
<th>$M_0 + 4$</th>
</tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

Xiao-Gang Wen
Long-range entangled quantum matter and a unification of forces
Second kind of edge:

\[
M_0 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 11111111111|00000000 \cdots \\
M_0 + 1 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 11111111110|10000000 \cdots \\
M_0 + 1 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 11111111102|00000000 \cdots \\
M_0 + 2 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 11111111110|01000000 \cdots \\
M_0 + 2 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 11111111101|10000000 \cdots \\
M_0 + 2 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 11111111020|10000000 \cdots \\
M_0 + 2 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 11111110202|00000000 \cdots 
\]

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<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>14</td>
</tr>
</tbody>
</table>
Greetings from The On-Line Encyclopedia of Integer Sequences!

Search: 1, 1, 2, 3, 5, 7, 11
Displaying 1-10 of 65 results found.

A000041  a(n) = number of partitions of n (the partition numbers).
(Formerly M0663 N0244)
1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4665, 5604, 6842, 8349, 10143, 12310, 14883, 17977, 21637, 26015, 31185, 37338, 44583, 53174, 63261, 75175, 89134 (list; graph; listen)
OFFSET 0,3
COMMENT Also number of nonnegative solutions to b+2c+3d+4e+...=n and the number of nonnegative solutions to 2c+3d+4e+...<=n. - Henry Bottomley (se16(AT)btinternet.com), Apr 17 2001
a(n) is also the number of conjugacy classes in the symmetric group S_n (and the number of irreducible representations of S_n).
Also the number of rooted trees with n+1 nodes and height at most 2.
Coincides with the sequence of numbers of nilpotent conjugacy classes in the Lie algebras gl(n). A006950, A015128 and this sequence together cover the nilpotent conjugacy classes in the classical A,B,C,D series of Lie algebras. - Alexander Elashvili, Sep 08 2003
a(n)=a(0)b(n)+a(1)b(n-2)+a(2)b(n-4)+... where b=A000009.
Number of distinct Abelian groups of order p^n, where p is prime (the number is independent of p). - Lekraj Beedassy (blekraj(AT)yahoo.com), Oct 16 2004
Local dancing rule $\rightarrow$ global dancing pattern

- Local dancing rules of a string liquid:
  1. Dance while holding hands (no open ends)
  2. $\Phi_{\text{str}}(\text{closed}) = \Phi_{\text{str}}(\text{open}), \Phi_{\text{str}}(\text{open}) = \Phi_{\text{str}}(\text{closed})$
  $\rightarrow$ Global dancing pattern $\Phi_{\text{str}}(\text{loop}) = 1$

- Local dancing rules of another string liquid:
  1. Dance while holding hands (no open ends)
  2. $\Phi_{\text{str}}(\text{closed}) = \Phi_{\text{str}}(\text{open}), \Phi_{\text{str}}(\text{open}) = -\Phi_{\text{str}}(\text{closed})$
  $\rightarrow$ Global dancing pattern $\Phi_{\text{str}}(\text{loop}) = (-)^{\# \text{ of loops}}$

- Two string-net condensations $\rightarrow$ two topological orders Levin-Wen 05

Xiao-Gang Wen Long-range entangled quantum matter and a unification of forces
Toric-code model – $\mathbb{Z}_2$ topological order, $\mathbb{Z}_2$ gauge theory

Dancing rules:

\[ \Phi_{\text{str}} (\bullet) = \Phi_{\text{str}} (\square), \quad \Phi_{\text{str}} (\square) = \Phi_{\text{str}} (\square)\]

- The Hamiltonian to enforce the dancing rules:

\[
H = -U \sum_I Q_I - g \sum_p F_p,
\]

\[ Q_I = \prod_{\text{legs of } I} \sigma_i^z, \quad F_p = \prod_{\text{edges of } p} \sigma_i^x \]

- Ground state wave function $\Phi(X) = \text{const.}$
Excitations

- The Hamiltonian is a sum of commuting operators
  \( [F_p, F_{p'}] = 0, [Q_I, Q_{I'}] = 0, [F_p, Q_I] = 0 \).

- \( F^2_p = Q^2_I = 1 \)

- Ground state \( |\psi_{\text{grnd}}\rangle \): \( F_p |\psi_{\text{grnd}}\rangle = Q_I |\psi_{\text{grnd}}\rangle = |\psi_{\text{grnd}}\rangle \),
  \( E_{\text{grnd}} = -2NU_{\text{cell}} - gN_{\text{cell}} \)

- Quasiparticle excitation energy gap \( \Delta^Q_p = 2U \), \( \Delta^F_p = 2g \)

Ground state degeneracy

- Identities \( \prod_I Q_I = 1, \prod_p F_p = 1 \).

- Number of independent quantum numbers \( F_p = \pm 1, Q_I = \pm 1 \) on torus:
  \( N_{\text{label}} = 4N_{\text{cell}}^2 / 4 \)

Number of states on torus:

- \( N_{\text{label}} = 2N_{\text{cell}}^2 \)

- \( H \) is a function of \( F_p, Q_I \). The degeneracy of any eigenstates is 4.

- The degenerate ground state subspace is a “code”, which has a large “code distance” of order \( L \) (the linear size of the system).
Dancing rules:

$$\Phi_{\text{str}} (\begin{array}{c}
\end{array}) = \Phi_{\text{str}} (\begin{array}{c}
\end{array}) , \quad \Phi_{\text{str}} (\begin{array}{c}
\end{array}) = - \Phi_{\text{str}} (\begin{array}{c}
\end{array})$$

- The Hamiltonian to enforce the dancing rules:

$$H = -U \sum_I Q_I - \frac{g}{2} \sum_p (F_p + h.c.),$$

$$Q_I = \prod_{\text{legs of } I} \sigma_i^z, \quad F_p = (\prod_{\text{edges of } p} \sigma_j^x)(\prod_{\text{legs of } p} i^{1-\sigma_j^z})$$

- Ground state wave function \( \Phi(X) = (-)^{X_c} \), where \( X_c \) is the number of loops in the string configuration \( X \).
Emergence of fractional spin/statistics

- Why electron carry spin-1/2 and Fermi statistics?
- Ends of strings are point-like excitations, which can carry spin-1/2 and Fermi statistics?

Fidkowski-Freedman-Nayak-Walker-Wang 06

\[ \Phi_{\text{str}}(\text{liquid}) = 1 \]

\[ \Phi_{\text{str}}(\text{liquid}) = \Phi_{\text{str}}(\text{liquid}) \]

360° rotation: \[ \bullet \rightarrow \circ \text{ and } \circ = \circ \rightarrow \bullet \]:

\[ R_{360^\circ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ \bullet + \circ \text{ has a spin } 0 \text{ mod } 1. \bullet - \circ \text{ has a spin } 1/2 \text{ mod } 1. \]

\[ \Phi_{\text{str}}(\text{liquid}) = (-1)^{\text{# of loops}} \]

\[ \Phi_{\text{str}}(\text{liquid}) = -\Phi_{\text{str}}(\text{liquid}) \]

360° rotation: \[ \bullet \rightarrow \circ \text{ and } \circ = -\circ \rightarrow -\bullet \]:

\[ R_{360^\circ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

\[ \bullet + i\circ \text{ has a spin } -1/4 \text{ mod } 1. \bullet - i\circ \text{ has a spin } 1/4 \text{ mod } 1. \]
Spin-statistics theorem

- (a) → (b) = exchange two string-ends.
- (d) → (e) = 360° rotation of a string-end.
- Amplitude (a) = Amplitude (e)
- Exchange two string-ends plus a 360° rotation of one of the string-end generate no phase.

→ Spin-statistics theorem
The string operators in the $Z_2$ state: the creation operator of topological quasiparticle

- Type-I string operator \( W_I = \prod_{\text{string}} \sigma_i^x \)
- Type-II string operator \( W_{II} = \prod_{\text{string}*} \sigma_i^z \)
- Type-III string operator \( W_{III} = \prod_{\text{string}} \sigma_i^x \prod_{\text{legs}} \sigma_i^z \)
- \([H, W_{\text{closed}}^I] = [H, W_{\text{closed}}^{II}] = [H, W_{\text{closed}}^{III}] = 0\).
- \( W_{\text{closed}}^I |\psi_{\text{grnd}}\rangle = W_{\text{closed}}^{II} |\psi_{\text{grnd}}\rangle = W_{\text{closed}}^{III} |\psi_{\text{grnd}}\rangle = |\psi_{\text{grnd}}\rangle \)
- \( Q_I \) and \( F_p \) are small closed string operators of type-I and type-II.
- Open string operators create topological excitations.
  Open string operators are hopping operators of topological excitations.
Statistics of ends of strings

- The statistics is determined by particle hopping operators Levin-Wen 03:

- An open string operator is a hopping operator of the ‘ends’. The algebra of the open string operator determines the statistics.

- For type-I strings: $t_{ba} = \sigma_1^x$, $t_{cb} = \sigma_3^x$, $t_{bd} = \sigma_2^x$
  
  We find $t_{bd} t_{cb} t_{ba} = t_{ba} t_{cb} t_{bd}$
  
  The ends of type-I strings are bosons

- For type-II strings: $t_{ba} = \sigma_1^x$, $t_{cb} = \sigma_3^x \sigma_4^z$, $t_{bd} = \sigma_2^x \sigma_3^z$
  
  We find $t_{bd} t_{cb} t_{ba} = -t_{ba} t_{cb} t_{bd}$
  
  The ends of type-II strings are fermions
Some remarks

- Number of topological type of quasiparticles = Number of degenerate ground states on torus
- Ground state degeneracy and strings
- Ground state degeneracy and string operator algebra
- Robust ground state degeneracy
Untwisted-string model: \[ H = -U \sum_I Q_I - g \sum_P B_P \]

\[ Q_I = \prod_{i \text{ next to } I} Z_i, \quad B_P = X_1 X_2 X_3 X_4 \]

Can get 3D fermions for free (almost)\(^{(1)}\) Levin & Wen 03

Just add a little twist

Twisted-string model: \[ H = U \sum_I Q_I - g \sum_P B_P \]

\[ B_P = X_1 X_2 X_3 X_4 Z_5 Z_6 \]
A pair of $Z_2$ charges is created by an open string operator which commute with the Hamiltonian except at its two ends. Strings cost no energy and is unobservable.

- In untwisted-string model – untwisted-string operator

$$X_{i_1} X_{i_2} X_{i_3} X_{i_4} \ldots$$

- In twisted-string model – twisted-string operator

$$\left( X_{i_1} X_{i_2} X_{i_3} X_{i_4} \ldots \right) \prod_{i \text{ on crossed legs of } C} Z_i$$
Twisted string operators commute \([W_1, W_2] = 0\)

\[
W_1 = (X_1 X_2 X_3 X_4 X_5 X_6 X_7) [Z_d Z_e Z_f] \\
W_2 = (X_h X_c X_5 X_4 X_3 X_d X_g) [Z_6 Z_e]
\]

- We also have \([W, Q_I] = 0\) for closed string operators \(W\), since \(W\) only create closed strings.
Statistics of ends of strings

- The statistics is determined by particle hopping operators Levin-Wen 03:

- An open string operator is a hopping operator of the ‘ends’. The algebra of the open string operator determine the statistics.

- For untwisted-string model: \( t_{ba} = X_2, t_{cb} = X_3, t_{bd} = X_1 \)
  
  We find \( t_{bd} t_{cb} t_{ba} = t_{ba} t_{cb} t_{bd} \)
  
  The ends of untwisted-string are bosons

- For twisted-string model: \( t_{ba} = Z_4 Z_1 X_2, t_{cb} = Z_5 X_3, t_{bd} = X_1 \)

  We find \( t_{bd} t_{cb} t_{ba} = -t_{ba} t_{cb} t_{bd} \)

  The ends of twisted-string are fermions

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Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
Origin of light = origin of Maxwell EM waves

Different orders (organizations) → different wave equations → different physical properties.

- Atoms in fluid have a random distribution → cannot resist shear deformations (which do nothing) → liquids do not have shapes

Wave Eq. → Euler Eq.
$$\partial_t^2 \rho - \partial_x^2 \rho = 0$$ One longitudinal mode

- Atoms in solid have a ordered lattice distribution → can resist shear deformations → solids have shapes

Wave Eq. → elastic Eq. $$\partial_t^2 u^i - C^{ijkl} \partial_x j \partial_x k u^l = 0$$ One longitudinal mode and two transverse modes
Origin of photons, gluons, electrons, quarks, etc

- Do all waves and wave equations emerge from some orders?

**Wave equations for elementary particles**

- Maxwell equation → Photons
  \[ \partial \times E + \partial_t B = \partial \times B - \partial_t E = \partial \cdot E = \partial \cdot B = 0 \]

- Yang-Mills equation → Gluons
  \[ \partial^\mu F^a_{\mu\nu} + f^{abc} A^b_{\mu} F^c_{\mu\nu} = 0 \]

- Dirac equation → Electrons/quarks (spin-1/2 fermions!)
  \[ [\partial_\mu \gamma^\mu + m] \psi = 0 \]

What orders produce the above waves? What are the origins of light (gauge bosons) and electrons (fermions)?

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Long-range entangled quantum matter and a unification of forces
Elementary or emergent?

- We used to think all orders are described by symmetry breaking, and different symmetry breaking orders indeed leads to different wave equations.
  - We just pick a particular symmetry breaking to produce the Maxwell equation.

- None of the symmetry breaking orders can produce:
  - Electromagnetic wave satisfying the Maxwell equation
  - Gluon wave satisfying the Yang-Mills equation
  - Electron wave satisfying the Dirac equation.

Two choices:
- Declare that photons, gluons, and electrons are elementary, and do not ask where do they come from.
- Declare that the symmetry breaking theory is incomplete. Maybe new orders beyond symmetry breaking can produce the Maxwell equation, Yang-Mills equation, and the Dirac equation.
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- Declare that photons, gluons, and electrons are elementary, and do not ask where do they come from.

- Declare that the symmetry breaking theory is incomplete. Maybe new orders beyond symmetry breaking can produce the Maxwell equation, Yang-Mills equation, and the Dirac equation.
Long range entanglements (closed strings) → emergence of electromagnetic waves (photons)

- Wave in superfluid state $|\Phi_{SF}\rangle = \sum_{\text{all position conf.}} |\psi\rangle$:
  - density fluctuations:
    - Euler eq.: $\partial_t^2 \rho - \partial_x^2 \rho = 0$
    - → Longitudinal wave

- Wave in closed-string liquid $|\Phi_{\text{string}}\rangle = \sum_{\text{closed strings}} |\psi\rangle$:
  - Wen 03, Levin-Wen 05
  - String density $E(x)$ fluctuations → waves in string condensed state.
  - Strings have no ends → $\partial \cdot E = 0$ → only two transverse modes.
  - Equation of motion for string density → Maxwell equation:
Long range entanglements (string nets) → Emergence of Yang-Mills theory (gluons)  

- If string has different types and can branch → string-net liquid → Yang-Mills theory
- Different ways that strings join → different gauge groups

A picture of our vacuum

Closed strings → $U(1)$ gauge theory
String-nets → Yang-Mills gauge theory

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XXZ Spin model on 2D Kagome lattice

- Only has nearest-neighbor and two-spin interactions (spin-1 spins):

\[ H = J_1 \sum (S_i^z)^2 + J_2 \sum S_i^z S_j^z - J_{xy} \sum (S_i^x S_j^x + S_i^y S_j^y) \]

Eigenstates of \( S^z \):

\[ S^z | \uparrow_z \rangle = | \uparrow_z \rangle \quad S^z | 0_z \rangle = 0 \quad S^z | \downarrow_z \rangle = - | \downarrow_z \rangle \]
Introduce $\Delta J = J_1 - J_2$ and rewrite

$$H = \frac{J_2}{2} \sum (S_1^z + S_2^z + S_3^z)^2 + \Delta J \sum (S_i^z)^2 - g \sum (S_1^+ S_2^- S_3^+ S_4^- S_5^+ S_6^- + h.c.)$$

When $\Delta J = g = 0$, the no string state and closed string states all have zero energy:

- **No string state:** $|0_z0_z0_z\ldots\rangle$
- **Closed-string state**
• The effect of $\Delta J$ term: String tension
• The effect of $g$ term: String hopping

When $\Delta J \ll g \ll J_2$, the ground state is a superposition of all closed-string states. Such a state is called *string-net condensed state* – a new state of matter that breaks no symmetries.

**Compare with some well known states**

• Crystal: Particles have a fixed regular positions.
• Superfluid (liquid): Particles have uncertain positions.
  Ground state = superposition of all particle positions.
• Plastic: Polymers have a fixed random configuration.
• String liquid: Strings have uncertain configurations.
  Ground state = superposition of all string-net configurations.
3D String-net condensation in cubic lattice

\[
H = U \sum_l Q_l^2 + J \sum (S_i^z)^2 - g \sum_p (B_p + h.c.)
\]

\[
Q_l = \sum_{i \text{ next to } l} S_i^z, \quad gB_p = gS_1^+S_2^-S_3^+S_4^-
\]

- \(U \sum_l Q_l^2\): only closed string states have low energies
- \(J \sum (S_i^z)^2\): string tension
- \(g \sum_p (B_p + h.c.)\): string hopping
Equation of motion approach $\rightarrow$ Maxwell equation

To understand the dynamics of $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{K}{2} \hat{x}^2$:

\[
\frac{d}{dt} \langle \hat{x} \rangle = \langle i[\hat{H}, \hat{x}] \rangle = \langle \hat{p} / m \rangle, \quad \frac{d}{dt} \langle \hat{p} \rangle = \langle i[\hat{H}, \hat{p}] \rangle = -\langle K \hat{x} \rangle
\]

Equation of motion of an oscillator.

**Emergence of Maxwell equation**

\[
B_p = e^{i \phi_p}, \quad S^z_i = E_i
\]

\[
\partial_t \langle S^z_i \rangle = \langle i[H, S^z_i] \rangle \sim i \langle \sum_{a=1,\ldots,4} B_{pa} - \text{h.c.} \rangle \sim \sum_{a=1,\ldots,4} \phi_{pa} \quad \rightarrow \quad \dot{E} = \partial \times B
\]

\[
\partial_t \langle B_p \rangle = \langle i[H, B_p] \rangle \sim i \langle \sum_{a=1,\ldots,4} S^z_{ia} B_p \rangle \sim i \sum_{a=1,\ldots,4} S^z_{ia} \quad \rightarrow \quad \dot{B} = \partial \times E
\]
Two gapped states, $|\psi(0)\rangle$ and $|\psi(1)\rangle$, are in the same phase iff they are connected by a local unitary (LU) evolution

$$|\psi(1)\rangle = P\left(e^{-i \int_{0}^{T} dt H(t)}\right)|\psi(0)\rangle$$

where $\tilde{H}(g) = \sum_i O_i(g)$ and $O_i(g)$ are local hermitian operators.

Hastings, Wen 05;
Bravyi, Hastings, Michalakis 10
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Hastings, Wen 05; Bravyi, Hastings, Michalakis 10

Any LU evolution can be described by a finite-depth quantum circuit – LU transformation:

$$|\Psi(1)\rangle = P \left( e^{-i \int_0^T dt H(t)} \right) |\Psi(0)\rangle = \prod \left( e^{-i \delta t H(t)} \right) |\Psi(0)\rangle$$
Pattern of long-range entanglements = topological order

For gapped systems with no symmetry:

- According to Landau theory, no symmetry to break
  → all systems belong to one trivial phase

For long-range entanglement, there are:

- Long range entangled (LRE) states → many phases
- Short range entangled (SRE) states → one phase

\[ |LRE\rangle \neq |\text{product state}\rangle = |SRE\rangle \]

Local unitary transformation

\[ g_1 g_2 \]

Phase transition

\[ \text{topological order} \]

\[ \text{different} \]

\[ \text{different} \]

Wen 1989

→ A classification by tensor category theory

Levin-Wen 05, Chen-Gu-Wen 2010

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Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
Pattern of long-range entanglements $\equiv$ topological order

**For gapped systems with no symmetry:**

- According to Landau theory, no symmetry to break $\rightarrow$ all systems belong to one trivial phase
- Thinking about entanglement: Chen-Gu-Wen 2010
  - There are long range entangled (LRE) states
  - There are short range entangled (SRE) states

\[ |LRE\rangle \neq \text{product state} = |SRE\rangle \]

\[ \text{local unitary transformation} \]

\[ \text{LRE state} \quad \text{SRE product state} \]
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  - There are short range entangled (SRE) states → one phase

\[ |\text{LRE}\rangle \neq |\text{product state}\rangle = |\text{SRE}\rangle \]

- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases
  = different patterns of long-range entanglements defined by the LU trans.

= different topological orders Wen 1989

Xiao-Gang Wen

Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
Topological order = pattern of long range entanglement
= equivalent class of LU transformations
How to label those equivalent classes?

We can use the wave function $\Phi$ itself to label the topological orders. But this is a many-many to one labeling scheme. Under the wave function renormalization, the wave function flows to simpler one within the same equivalent class.

Use the fixed-point wave function: $\Phi \rightarrow \Phi_{\text{fix}}$ to label topological order. Hopefully $\Phi_{\text{fix}}$ can give us a one-to-one labeling of topological order, and a classification of topological order.
Labeling topological orders

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The concept of support space

Divide the system into two regions $A$ and $B$. The total Hilbert space $\mathcal{V}_{tot} = \bigotimes_i \mathcal{V}_i = \mathcal{V}_A \otimes \mathcal{V}_B$

A state can be written as $|\psi\rangle = \sum_{\alpha} |\psi_A(\alpha)\rangle \otimes |\psi_B(\alpha)\rangle$

Support space $\mathcal{V}_A^{sp}$ on $A = \text{The Hilbert space spanned by } |\psi_A(\alpha)\rangle$

The dimension of the support space is less than the total Hilbert space $\mathcal{V}_A$ in $A$: $\mathcal{V}_A^{sp} \subset \mathcal{V}_A$

The concept of generalized unitary transformation

A generalized unitary transformation $U : \mathcal{V}_A \rightarrow \mathcal{V}_A^{sp}$ shrinks the degrees of freedom in $A$ without losing any quantum information. It generates a wave function renormalization:

Vidal 07; Jordan, Orus, Vidal, Verstraete, Cirac 08; Jiang, Weng, Xiang 09; Gu, Levin, Wen 09

Long-range entangled quantum matter and a unification of forces
Classify topological orders with fixed-point LU trans.

But using fixed-point wave function $\Phi_{\text{fix}}$ to label topological orders has one problem:
as we perform wave function renormalization, the number of degrees of freedom and size/shape of lattice are changing. The fixed-point wave function $\Phi_{\text{fix}}$ can never be fixed.

The concept of fixed-point state

- A fixed-point state is not one wave function, but a family of wave functions, $\Phi_n$, one wave function of each size/shape of lattice.
- "Fixed point" does not mean the fixed wave function. It means a fixed relation between those wave functions, fixed-point local unitary (LU) transformation, $U^\infty \Phi(\text{lattice-2}) = U^\infty \Phi(\text{lattice-1})$, $\Phi(\text{lattice-3}) = U^\infty \Phi(\text{lattice-2})$.

Topological orders are classified by fixed-point local unitary transformations.
Classify topological orders with fixed-point LU trans.

But using fixed-point wave function $\Phi_{\text{fix}}$ to label topological orders has one problem: as we perform wave function renormalization, the number of degrees of freedom and size/shape of lattice are changing. The fixed-point wave function $\Phi_{\text{fix}}$ can never be fixed.

$$\Phi_1 \xrightarrow{U_1} \Phi_2 \xrightarrow{U_2} \Phi_3 \xrightarrow{U_3}$$

The concept of fixed-point state

- A fixed-point state is not one wave function, but a family of wave functions, $\Phi_n$, one wave function of each size/shape of lattice.
- “Fixed point” does not mean the fixed wave function. It means a fixed relation between those wave functions, 
  - fixed-point local unitary (LU) transformation, $U_\infty$

  $$\Phi(\text{lattice-2}) = U_\infty \Phi(\text{lattice-1})$$

  $$\Phi(\text{lattice-3}) = U_\infty \Phi(\text{lattice-2})$$

Topological orders are classified by fixed-point local unitary transformations
To find fixed-point LU transformations, we need to first have some understanding of (or make some assumptions to) fixed-point states.
The structure of fixed-point states

To find fixed-point LU transformations, we need to first have some understanding of (or make some assumptions to) fixed-point states.

**Graphic state:**

- Fixed-point wave functions are defined on graphs, with \( N + 1 \) states on links and \( N_v \) states on vertices:

\[
\begin{align*}
\alpha &= \beta = \gamma = \lambda \\
\alpha &= 1, \ldots, N_v \\
\beta &= \ldots \\
\gamma &= \ldots \\
\lambda &= \ldots \\
i &= 0, \ldots, N
\end{align*}
\]

- Support space and support dimension with boundary:

\[
\Phi(\alpha, \beta, m, i, j, k, l; \Gamma) = \psi_{i, j, k, l, \Gamma}(\alpha, \beta, m)
\]

- Support space on \( \alpha\beta m \):

\[
V_{ijkl}^* = \{ \psi_{i, j, k, l, \Gamma}(\alpha, \beta, m) | \text{fix} \}
\]

- Support dim.:

\[
D_{ijkl}^* = \dim V_{ijkl}^* \text{ for the region bounded by } ijkl.
\]
The structure of fixed-point states

To find fixed-point LU transformations, we need to first have some understanding of (or make some assumptions to) fixed-point states.

**Graphic state:**
- Fixed-point wave functions are defined on graphs, with \( N + 1 \) states on links and \( N_v \) states on vertices:

\[
\alpha = \beta = \gamma = \lambda
\]

- Support space and support dimension with boundary:

\[
\Phi \left( \begin{array}{c}
\alpha \\
\beta \\
m \\
\gamma
\end{array} \right) = \Phi(\alpha, \beta, m, i, j, k, l; \Gamma) = \psi_{i,j,k,l,\Gamma}(\alpha, \beta, m)
\]

Support space on \( \alpha \beta m \): \( V_{ijkl} = \{\psi_{i,j,k,l,\Gamma}(\alpha, \beta, m) | \text{fix } ijk \}, \text{ vary } \Gamma \} \)

Support dim.: \( D_{ijkl} = \text{dim} V_{ijkl} \) for the region bounded by \( ijk \).
The support dimensions of tree graphs

• The support dimension of the $\Phi(\alpha \beta \chi_{ijk})$ on a region bounded by links $i, j, k$: $D_{ijk} \leq N_v$
  $\rightarrow$ shrink the range $\alpha = 1, \ldots, N_{ijk} = D_{ijk}$ (which depends on $ijk$).

• **Saturation condition**: For $\Phi = (\alpha \beta \chi_{ijkl})$:
  
  The support dimension $D_{ijkl}^*$
  $= \text{The number of } \alpha \beta m = \sum_m N_{jm}^* N_{kl}^*$

• Similar saturation condition for any “tree” region
  
  $\Phi = (\alpha \beta \chi_{ijkl})$
  $D_{ijklp}^* = \sum_{m,n} N_{jm}^* N_{mn}^* k N_{np}^* l$
The first kind of wave function renormalization: F-move

- The fixed-point wave functions are related by a fixed LU trans.:

\[ \Phi(\text{graph-2}) = U_\infty \Phi(\text{graph-1}) \]

- \( \Phi = \left( \begin{array}{c} i \atop \alpha \beta \atop \delta \neq \gamma \atop m \atop l \end{array} \right) \) and \( \Phi = \left( \begin{array}{c} i \atop \chi \delta \atop \gamma \neq \delta \atop n \atop l \end{array} \right) \) have the same support

\[
\sum_m N_{jm}^* N_{km}^* = \sum_n N_{kn}^* N_{nl}^* \equiv N_{ijkl}^*
\]

- The two wave functions are related by a LU trans. \( \text{Leven, Wen, 04} \)

F-move: \( \Phi \left( \begin{array}{c} i \atop \alpha \beta \atop \delta \neq \gamma \atop m \atop l \end{array} \right) = \sum_{n=0}^{N} \sum_{\chi=1}^{N_{kn}^*} \sum_{\delta=1}^{N_{nl}^*} F_{i,j}^{m,\alpha,\beta} \Phi \left( \begin{array}{c} i \atop \chi \delta \atop \gamma \neq \delta \atop n \atop l \end{array} \right) \)

The matrix \( F_{k,l}^{ij} \to (F_{k,l}^{ij})_n^{m,\alpha,\beta} \) is unitary and has a dimension \( N_{ijkl}^* \).
The pentagon identity

\[
\Phi(ijklmp) = \sum_{q,\delta,\epsilon} F_{lpq,\delta\epsilon}^{mkn,\beta\chi} \Phi(ijklmp) = \sum_{q,\delta,\epsilon;s,\phi,\gamma} F_{lpq,\delta\epsilon}^{mkn,\beta\chi} F_{ijm,\alpha\epsilon}^{qps,\phi\gamma} \Phi(ijklmp)
\]

\[
\Phi(ijklmp) = \sum_{t,\eta,\varphi} F_{ijm,\alpha\beta}^{kmnt,\eta\varphi} \Phi(ijklmp) = \sum_{t,\eta,\varphi;s,\kappa,\gamma} F_{ijm,\alpha\beta}^{kmnt,\eta\varphi} F_{litnq,\varphi\chi}^{kps,\kappa\gamma} \Phi(ijklmp)
\]

\[
\Phi(ijklmp) = \sum_{t,\eta,\kappa;\varphi,s,\kappa,\gamma;q,\delta,\phi} F_{ijm,\alpha\beta}^{kmnt,\eta\varphi} F_{litnq,\varphi\chi}^{kps,\kappa\gamma} F_{jkt,\eta\kappa}^{lqs,\delta\phi} \Phi(ijklmp)
\]

The two paths should lead to the same LU trans.:

\[
\sum_{t,\eta,\varphi,\kappa} F_{ijm,\alpha\beta}^{kmnt,\eta\varphi} F_{litnq,\varphi\chi}^{kps,\kappa\gamma} F_{jkt,\eta\kappa}^{lqs,\delta\phi} = \sum_{\epsilon} F_{lpq,\delta\epsilon}^{mkn,\beta\chi} F_{ijm,\alpha\epsilon}^{qps,\phi\gamma}
\]

Such a set of non-linear algebraic equations is the famous pentagon identity.

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Long-range entangled quantum matter and a unification of...
The second kind of wave function renormalization: P-move

- First way: \[ \begin{array}{c}
\begin{array}{c}
\alpha \\
i
\end{array}
\end{array} \rightarrow
\begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} \]
which reduce the degrees of freedom.

But notice that, through the F-move,

- Second way:

\[ \begin{array}{c}
\begin{array}{c}
\alpha \\
i
\end{array}
\end{array} \rightarrow
\begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} \]

Levin, Wen, 04  
or  
Koenig, Reichardt, Vidal, 09

The support dimension of \( \Phi \):

\[ D_{ii'\star} = \delta_{ii'} \]

\[ \Phi \left( \begin{array}{c}
\begin{array}{c}
\beta \\
i
\end{array}
\end{array} \right) = P_{ij,\alpha\beta}^{k} \Phi \left( \begin{array}{c}
\begin{array}{c}
\iota \\
i
\end{array}
\end{array} \right) , \quad \sum_{\alpha=1}^{N_{ji}^{\star}} \sum_{\beta=1}^{N_{ji}^{\star}} P_{ij,\alpha\beta}^{k} (P_{ij,\alpha\beta}^{k})^{\star} = 1 \]
Consistent conditions between $F_{ijm,\alpha\beta}^{kln,\chi\delta}$ and $P_{ij}^{kj,\alpha\beta}$

From

$$\Phi (\begin{array}{c}
\alpha \\
\eta \\
\beta \\
l \\
\end{array}) = \sum_{n=0}^{N} \sum_{\chi,\delta} F_{ijm,\alpha\beta}^{kln,\chi\delta} \Phi (\begin{array}{c}
\beta \\
\delta \\
\eta \\
l \\
\end{array})$$

$$\rightarrow P_{ij}^{jp,\alpha\eta} \delta_{im} \Phi (\begin{array}{c}
\beta \\
l \\
\end{array}) = \sum_{n,\chi,\delta} F_{ijm,\alpha\beta}^{kln,\chi\delta} P_{k^*}^{jp,\chi\eta} \delta_{kn} \Phi (\begin{array}{c}
\delta \\
l \\
\end{array})$$

we find more non-linear equation

$$P_{ij}^{jp,\alpha\eta} \delta_{im} \delta_{\beta\delta} = \sum_{\chi=1}^{N_{kjk^*}} F_{ijm,\alpha\beta}^{klk,\chi\delta} P_{k^*}^{jp,\chi\eta}$$

for all $k, i, l$ satisfying $N_{kil^*} > 0$
A classification of topological orders

The data $N_{ijk}, F_{kln, \chi \delta}^{ijm, \alpha \beta}, P_{i}^{kj, \alpha \beta}$ classify the fixed-point LU transformation and topological orders. They satisfy

$$
\sum_{m} N_{jim}^{*} N_{kml}^{*} = \sum_{n} N_{kjn}^{*} N_{l*ni}^{*},
$$

$$(F_{kln, \chi \delta}^{ijm, \alpha \beta})^{*} = F_{jkn, \chi \delta}^{i*im*}, \beta \alpha, n, \chi, \delta
$$

$$
\sum_{n, \chi, \delta} F_{kln, \chi \delta}^{ijm', \alpha' \beta'} \left(F_{kln, \chi \delta}^{ijm, \alpha \beta}\right)^{*} = \delta_{m \alpha \beta, m' \alpha' \beta'},
$$

$$
\sum_{t, \eta, \varphi, \kappa} F_{knt, \eta \varphi}^{ijm, \alpha \beta} F_{lps, \kappa \gamma}^{itn, \varphi \chi} F_{lsq, \delta \phi}^{jkt, \eta \kappa} = \sum_{\epsilon} F_{lps, \delta \epsilon}^{mkn, \beta \chi} F_{qps, \phi \gamma}^{ijm, \alpha \epsilon},
$$

$$
N_{kii}^{*} N_{j*jk}^{*} = \sum_{\alpha=1}^{\beta=1} P_{i}^{kj, \alpha \beta} (P_{i}^{kj, \alpha \beta})^{*} = 1,
$$

$$
P_{i}^{kj, \alpha \beta} = \sum_{m, \lambda, \gamma, l, \nu, \mu} F_{i*im*, \lambda \gamma}^{jjk*, \beta \alpha} F_{i*im*, \lambda \gamma}^{m*ijl, \nu \mu} P_{i}^{lm, \mu \nu},
$$

$$
P_{i}^{jp, \alpha \eta} \delta_{im} \delta_{\beta \delta} = \sum_{\chi} F_{klk, \chi \delta}^{ijm, \alpha \beta} P_{k*}^{jp, \chi \eta} \text{ for all } k, i, l \text{ with } N_{kil*} > 0.
$$
Spherical fusion category and local Hamiltonian

- The data $N_{ijk}, F^{ijm,\alpha\beta}, P^{kj,\alpha\beta}$ describe a spherical fusion category
- From the spherical fusion category $\rightarrow$ wave functions on any graph: $\text{Graph-0} \xrightarrow{F_{01}} \text{Graph-1} \xrightarrow{F_{12}} \text{Graph-2} \ldots \xrightarrow{F_{n0}} \text{Graph-0}$

A condition on states on Graph-0: $P_p \psi_0 = \psi_0$

$\rightarrow$ A unique state on sphere, a few states on genus-$g$ surface.

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Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution.
The data $N_{ijk}, F_{ijm,\alpha\beta}, P_{kj,\alpha\beta}$ describe a spherical fusion category.

From the spherical fusion category $\rightarrow$ wave functions on any graph: 

$$\text{Graph-0} \xrightarrow{F_{01}} \text{Graph-1} \xrightarrow{F_{12}} \text{Graph-2} \ldots \xrightarrow{F_{n0}} \text{Graph-0}$$

A condition on states on Graph-0: $P_p \Psi_0 = \Psi_0$ $\rightarrow$ A unique state on sphere, a few states on genus-$g$ surface.

- $H$ is a local hermitian operator (a sum of commuting projectors)
  $$H = \sum (1 - P_p) + \sum Q_I = \sum B_p + \sum Q_I$$ $\rightarrow$ local Hamiltonian.
Simple solutions of the non-linear equations

We choose $N_{000} = N_{110} = N_{101} = N_{011} = 1$, other $N_{ijk} = 0$, and find two sets of solutions

\[
F_{000}^{000} \langle \bar{\Phi} \rangle = 1
\]
\[
F_{111}^{000} \langle \bar{\Phi} \rangle = (F_{100}^{011} \bar{\Phi} \Phi^*) = (F_{010}^{101} \bar{\Phi} \Phi^*) = F_{001}^{110} \langle \bar{\Phi} \Phi \rangle = 1
\]
\[
F_{011}^{011} \langle \bar{\Phi} \rangle = (F_{101}^{101} \bar{\Phi} \Phi^*) = 1
\]
\[
F_{110}^{110} \langle \bar{\Phi} \rangle = \eta = \pm 1
\]

Both solutions are closed-string states:

- $Z_2$ state: $\eta = 1$: $\Phi(\text{loops}) = 1$
- Semion state: $\eta = -1$: $\Phi(\text{loops}) = (-1)^{\# \text{ of loops}}$

Freedman, Nayak, Shtengel, Walker, Wang, 04; Levin, Wen, 04

• $Z_2$ state: $\eta = 1$: $\Phi(\text{loops}) = 1$
• Semion state: $\eta = -1$: $\Phi(\text{loops}) = (-1)^{\# \text{ of loops}}$
We choose
\( N_{000} = N_{110} = N_{101} = N_{011} = N_{111} = 1 \), other \( N_{ijk} = 0 \), and find only one set of solutions:

\[
F_{000}^{000} = 1
\]
\[
F_{111}^{000} = (F_{100}^{011} F_{101}^{011})^* = (F_{010}^{101} F_{101}^{010})^* = F_{001}^{110} = 1
\]
\[
F_{011}^{011} = (F_{101}^{101} F_{101}^{101})^* = 1
\]
\[
F_{111}^{111} = (F_{111}^{101} F_{111}^{101})^* = F_{011}^{111} F_{011}^{111} = (F_{101}^{111} F_{101}^{111})^* = 1
\]
\[
F_{110}^{110} = \gamma
\]
\[
F_{111}^{110} = (F_{110}^{111} F_{110}^{111})^* = \sqrt{\gamma}
\]
\[
F_{111}^{111} = -\gamma, \quad \gamma = (\sqrt{5} - 1)/2
\]

\( N = 1 \) string-net state = Fibonacci state Freedman, Nayak, Shtengel, Walker, Wang, 04; Levin, Wen, 04
\( \mathbf{Z}_2 \) theory

- \( N = 1, \, \delta_{000} = \delta_{110} = 1, \, \delta_{100} = 0 \) (only closed strings), \( x \times x = 1 \)

\( \mathbf{F}^{ijm}_{kln} \) leads to

\[
\Phi \left( \begin{array}{c}
\text{Z} \\
\text{Z} \\
\text{X}
\end{array} \right) = \Phi \left( \begin{array}{c}
\text{Z} \\
\text{X}
\end{array} \right), \quad \Phi \left( \begin{array}{c}
\text{Z} \\
\text{X} \\
\text{Z}
\end{array} \right) = \Phi \left( \begin{array}{c}
\text{Z} \\
\text{Z}
\end{array} \right)
\]

- The Hamiltonian

\[
H_{\text{str}} = \sum Q_l + \sum B_p = \sum (1 - \prod_{\text{legs of } l} Z_i) + \sum \left[ 1 - \left( \prod_{\text{edges of } p} X_j \right) \right]
\]

- Ground state wave function \( \Phi(L) = 1 \),

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Doubled semion theory

- $N = 1$, $\delta_{000} = \delta_{110} = 1$, $\delta_{100} = 0$ (only closed strings), $x \times x = 1$

$F_{ijm}^{klm}$ leads to

$$\Phi(\square) = - \Phi(\square), \quad \Phi(\square \swarrow \searrow) = - \Phi(\square \swarrow \searrow)$$

- The Hamiltonian

$$H_{\text{str}} = \sum_{I} (1 - \prod_{\text{legs of } I} Z_i) + \sum_{p} \left[ 1 - \left( \prod_{\text{edges of } p} X_j \right) (-)^{\# \text{ of str to } p} \right]$$

$$\# \text{ of str to } p = \sum_{\text{legs of } p} (1 - Z_i)/2$$

- Ground state wave function $\Phi(L) = (-)^{L_c}$, where $L_c$ is the number of loops in the string configuration $L$
The data $N_{ijk}, F^{ijm, \alpha\beta}_{kln, \chi\delta}, P^{kj, \alpha\beta}_i$ describe a spherical fusion category $H = \sum Q_I + \sum B_p$.

Ground state: $B_p \Phi_0 = Q_I \Phi_0 = 0$ for every $i$.

Local excited state: $B_p \Phi_0 = Q_I \Phi_0 = 0$ for most $i$, but $B_p \Phi_0 \neq 0$, $Q_I \Phi_0 \neq 0$ for a few $i$.

A local excited state behave like a particle $\rightarrow$ a quasiparticle.

Quasiparticles can fuse $a \otimes b \rightarrow c + d...$: $\Phi_{a,b} = \Phi_c + \Phi_d...$

Quasiparticles can braid $a \leftrightarrow b$: $\Phi_{a,b} \rightarrow e^{i\theta} \Phi_{a,b}$

Quasiparticles have non-Abelian statistics described by a MTC.

Spherical fusion category $\rightarrow$ MTC that describes the quasiparticles

The MTC = The Drinfeld center of the spherical fusion category.
MTC data: ground state degeneracy

- Any ground state on any graph on torus can be reduced to a ground state on a simple graph on torus.

- The ground state deg. on torus satisfies $D_{tor} \leq \sum_{ijk} N_{ik}^* j N_{i^* k^*}$

- For the $Z_2$ state: $D_{tor} \leq 4$.
  - For the semion state: $D_{tor} \leq 4$.
  - For the Fibonacci state: $D_{tor} \leq 5$.
MTC data: ground state degeneracy

- Any ground state on any graph on torus can be reduced to a ground state on a simple graph on torus.

- The ground state deg. on torus satisfies $D_{\text{tor}} \leq \sum_{ijk} N_{ik} j N_{i^*} k_j^*$.

- For the $Z_2$ state: $D_{\text{tor}} \leq 4$.
  - For the semion state: $D_{\text{tor}} \leq 4$.
  - For the Fibonacci state: $D_{\text{tor}} \leq 5$.

- After considering the Hamiltonian on the torus, we find
  - for the $Z_2$ state: $D_{\text{tor}} = 4$.
  - for the semion state: $D_{\text{tor}} = 4$.
  - for the Fibonacci state: $D_{\text{tor}} = 4$. 
T-matrix and Dehn twist

\[ \Phi_{\alpha\beta}^{\text{fix}} \rightarrow \Phi_{\alpha\beta}^{\text{fix}} \]

\[ = \sum_{l,\chi\delta} F_{ijk,\alpha\beta}^{i*jl*,\chi\delta} \Phi_{\chi\delta}^{\text{fix}} \]

\[ = \sum_{l,\chi\delta} F_{ijk,\alpha\beta}^{i*jl*,\chi\delta} \Phi_{\chi\delta}^{\text{fix}} \]

\[ = \sum_{i'j'k',\alpha'\beta'} \delta_{ik'} \delta_{jj'} F_{iji'j'i'^*,\alpha'\beta'}^{\alpha'\beta'} \Phi_{\alpha'\beta'}^{\text{fix}} \]

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Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
T-matrix for the $Z_2$ state

\[
\Phi_{\text{fix}} \left( \begin{array}{c} 000 \\ 000 \end{array} \right) = F^{000} \Phi_{\text{fix}} \left( \begin{array}{c} 000 \\ 000 \end{array} \right), \quad T = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right)
\]

\[
\Phi_{\text{fix}} \left( \begin{array}{c} 101 \\ 101 \end{array} \right) = F^{101} \Phi_{\text{fix}} \left( \begin{array}{c} 101 \\ 101 \end{array} \right), \quad \sim \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right)
\]

\[
\Phi_{\text{fix}} \left( \begin{array}{c} 011 \\ 011 \end{array} \right) = F^{011} \Phi_{\text{fix}} \left( \begin{array}{c} 011 \\ 011 \end{array} \right), \quad U = \left( \begin{array}{cccc} 1 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array} \right)
\]

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Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
S-matrix for the $Z_2$ state

\[ \Phi_{\text{fix}} \begin{pmatrix} 000 \end{pmatrix} = F_{000}^{000} \Phi_{\text{fix}} \begin{pmatrix} 000 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ \Phi_{\text{fix}} \begin{pmatrix} 000 \end{pmatrix} = F_{101}^{101} \Phi_{\text{fix}} \begin{pmatrix} 101 \end{pmatrix}, \quad \sim \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \]

\[ \Phi_{\text{fix}} \begin{pmatrix} 000 \end{pmatrix} = F_{011}^{011} \Phi_{\text{fix}} \begin{pmatrix} 011 \end{pmatrix}, \quad U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \]
T-matrix for the double semion state

\[ \Phi_{\text{fix}} \left( \begin{array}{c} 000 \\ \end{array} \right) = F_{000}^{000} \Phi_{\text{fix}} \left( \begin{array}{c} 000 \\ \end{array} \right), \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \]

\[ \Phi_{\text{fix}} \left( \begin{array}{c} 101 \\ \end{array} \right) = F_{101}^{101} \Phi_{\text{fix}} \left( \begin{array}{c} 101 \\ \end{array} \right), \quad \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \]

\[ \Phi_{\text{fix}} \left( \begin{array}{c} 110 \\ \end{array} \right) = F_{011}^{011} \Phi_{\text{fix}} \left( \begin{array}{c} 011 \\ \end{array} \right), \quad U = \begin{pmatrix} i \sqrt{2} & i \sqrt{2} & 0 & 0 \\ 0 & -1 \sqrt{2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \]
S-matrix for the double semion state

\[ \Phi_{\text{fix}} \begin{pmatrix} 000 \\ \end{pmatrix} = F^{000}_{\text{fix}} \Phi_{\text{fix}} \begin{pmatrix} 000 \\ \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

\[ \Phi_{\text{fix}} \begin{pmatrix} 101 \\ \end{pmatrix} = F^{101}_{\text{fix}} \Phi_{\text{fix}} \begin{pmatrix} 110 \\ \end{pmatrix}, \quad \sim \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \]

\[ \Phi_{\text{fix}} \begin{pmatrix} 110 \\ \end{pmatrix} = F^{011}_{\text{fix}} \Phi_{\text{fix}} \begin{pmatrix} 101 \\ \end{pmatrix}, \quad U = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \]
MTC data: degenerate vector space of quasiparticle

- An end of an open string represents a quasiparticle.
- A state with $n$ ends of strings can be reduced to a tree graph.
MTC data: degenerate vector space of quasiparticle

• An end of an open string represents a quansiparticle.
• A state with $n$ ends of strings can be reduced to a tree graph.

\[
\begin{array}{cccccccc}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\end{array}
\]

$Z_2$ state:

• One non-degenerate $2n$-quasiparticle state.
MTC data: degenerate vector space of quasiparticle

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$Z_2$ state:

- One non-degenerate $2n$-quasiparticle state.

Fibonacci:

- $F_{n-2}$ degenerate $n$-quasiparticle states, even if we fix the position of the all the $n$ quasiparticles. $F_n = F_{n-1} + F_{n-2}$, $F_0 = F_1 = 1$.
- Ends of open strings $\rightarrow$ particles with non-Abelian statistics
Quasi-particles and long string operators

- Long-string operator $W^\alpha$ commute with the Hamiltonian $[W^\alpha, H_{str}] = 0$. $B^s_p$ is a special case of $W^\alpha_{str}$ (short string).
- Definition of (simple) long-string operator $W^\alpha_{str}$ (on fattened lattice)

\[
\begin{align*}
\langle \alpha^\alpha | = | \alpha^s \rangle, \\
\langle \alpha_i | = \sum_j \omega^j_{s,i} | j_s i^s \rangle, \\
\langle \alpha_i^* | = \sum_j \bar{\omega}^j_{s,i} | j^s i^s \rangle.
\end{align*}
\]

- Conditions on $\omega^j_{s,i}$ and $\bar{\omega}^j_{s,i}$

\[
\begin{align*}
| j^i_k \rangle &= | j^i_k \rangle, \\
| i^s_j \rangle &= | i^s_j \rangle
\end{align*}
\]

\[
\bar{\omega}^m_{s,j} F_{s^*n^*}^{k^*l^*} \omega^n_{s,i} \frac{V_j V_s}{V_m} = \sum_{n=0}^{N} F_{s^*n^*}^{j^*k^*} \omega^n_{s,k} F_{k^*l^*m^*}^{j^*n^*}
\]

\[
\bar{\omega}^j_{s,i} = \sum_{k=0}^{N} \omega^k_{s,j^*} F_{i^*s^*j^*}^{k^*n^*}.
\]
Z₂ Long string operators

\[ |1\rangle = |1\rangle, \quad |2\rangle = |2\rangle, \quad |3\rangle = |3\rangle, \quad |4\rangle = |4\rangle, \quad |\rangle = |\rangle, \quad |\rangle = |\rangle, \quad |\rangle = |\rangle, \quad |\rangle = |\rangle. \]

Note that \( W^4 = W^2 W^3 \). There are 3+1 kinds of quasiparticles.

\[ \langle 3 | = (-1)^2 \langle 3 | \Rightarrow \langle 3 | = \bigcirc = \bigcirc \]

- Mutual statistics \( \theta_{\alpha\beta} \) and statistics \( \theta_{\alpha} \):

\[
e^{i\theta_{\alpha\beta}} = \frac{\langle \Phi | \beta \bigotimes \alpha | \Phi \rangle}{\langle \Phi | \beta \bigotimes \alpha | \Phi \rangle}, \quad e^{i\theta_{\alpha}} = \frac{\langle \Phi | \alpha | \Phi \rangle}{\langle \Phi | \alpha | \Phi \rangle}.
\]

\[
\left( e^{i\theta_{\alpha\beta}} \right) = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}, \quad \left( e^{i\theta_{\alpha}} \right) = \begin{pmatrix}
1 \\
1 \\
1 \\
-1
\end{pmatrix}.
\]
Effective theory

$Z_2$ gauge theory = $U(1) \times U(1)$ Chern-Simons theory

$$L = \frac{1}{4\pi} K_{IJ} a_I a_J \partial_\nu a_J \epsilon^{\mu\nu\lambda}, \quad K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

which produces the same quasiparticle (mutual) statistics:

\[
\begin{pmatrix}
\text{type-1} \\
\text{type-2} \\
\text{type-3} \\
\text{type-4}
\end{pmatrix} = 
\begin{pmatrix}
\text{trivial} \\
Z_2\text{-charge} \\
Z_2\text{-vortex} \\
Z_2\text{-charge-vortex}
\end{pmatrix} = 
\begin{pmatrix}
\text{trivial} \\
a_1\text{-charge} \\
a_2\text{-charge} \\
a_1-a_2\text{-charge}
\end{pmatrix} = 
\begin{pmatrix}
\text{Boson} \\
\text{Boson} \\
\text{Boson} \\
\text{Fermion}
\end{pmatrix}
\]
Double-semion Long string operators

\[
\left| \begin{array}{c}
1 \\
2
\end{array} \right\rangle = \left| \begin{array}{c}
1 \\
2
\end{array} \right\rangle , \quad \left| \begin{array}{c}
1 \\
3
\end{array} \right\rangle = \left| \begin{array}{c}
1 \\
3
\end{array} \right\rangle ,
\left| \begin{array}{c}
1 \\
4
\end{array} \right\rangle = \left| \begin{array}{c}
1 \\
4
\end{array} \right\rangle ,
\left| \begin{array}{c}
1 \\
5
\end{array} \right\rangle = \left| \begin{array}{c}
1 \\
5
\end{array} \right\rangle ,
\left| \begin{array}{c}
1 \\
6
\end{array} \right\rangle = \left| \begin{array}{c}
1 \\
6
\end{array} \right\rangle ,
\left| \begin{array}{c}
1 \\
7
\end{array} \right\rangle = \left| \begin{array}{c}
1 \\
7
\end{array} \right\rangle ,
\left| \begin{array}{c}
1 \\
9
\end{array} \right\rangle = \left| \begin{array}{c}
1 \\
9
\end{array} \right\rangle .
\]

There are 3+1 kinds of quasiparticles.

\[
\left| \begin{array}{c}
1 \\
2
\end{array} \right\rangle = \left| \begin{array}{c}
1 \\
2
\end{array} \right\rangle , \quad \left| \begin{array}{c}
1 \\
3
\end{array} \right\rangle = i \left| \begin{array}{c}
1 \\
3
\end{array} \right\rangle , \quad \left| \begin{array}{c}
1 \\
4
\end{array} \right\rangle = \left| \begin{array}{c}
1 \\
4
\end{array} \right\rangle , \quad \left| \begin{array}{c}
1 \\
5
\end{array} \right\rangle = -i \left| \begin{array}{c}
1 \\
5
\end{array} \right\rangle .
\]

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\[
e^{i \theta_{\alpha\beta}} = \frac{\langle \Phi \left| \begin{array}{c}
\beta \\
\alpha
\end{array} \right| \Phi \rangle}{\langle \Phi \left| \begin{array}{c}
\beta \\
\alpha
\end{array} \right| \Phi \rangle}, \quad e^{i \theta_{\alpha}} = \frac{\langle \Phi \left| \begin{array}{c}
\alpha
\end{array} \right| \Phi \rangle}{\langle \Phi \left| \begin{array}{c}
\alpha
\end{array} \right| \Phi \rangle}.
\]

\[
\left( e^{i \theta_{\alpha\beta}} \right) = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}, \quad \left( e^{i \theta_{\alpha}} \right) = \begin{pmatrix}
1 \\
i \\
1 \\
1
\end{pmatrix}.
\]
Double semion effective theory

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$$\begin{pmatrix} \text{type-1} \\ \text{type-2} \\ \text{type-3} \\ \text{type-4} \end{pmatrix} = \begin{pmatrix} \text{trivial} \\ a_1\text{-charge} \\ a_1-a_2\text{-charge} \\ a_2\text{-charge} \end{pmatrix} = \begin{pmatrix} \text{Boson} \\ \text{Semion} \\ \text{Boson} \\ \text{Semion} \end{pmatrix}$$
Gapped phases w/ symmetry → SET and SPT phases

- **there are LRE symmetric states** → Symm. Enriched Topo. phases
  - 100s symm. spin liquid through the PSG of topo. excit. Wen 02
  - 8 trans. symm. enriched $Z_2$ topo. order in 2D, 256 in 3D Kou-Wen 09
  - Many symm. $Z_2$ spin liquid through $[H^2(SG, Z_2)]^2 \times$ Hermele 12
  - Classify SET phases through $H^3[SG \times GG, U(1)]$ Ran 12

Xiao-Gang Wen
Long-range entangled quantum matter and a unification of forces
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- **there are** **SRE symmetric states** → one phase

---

**Xiao-Gang Wen**

Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
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  We may call them symmetry protected trivial (SPT) phase

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• there are SRE symmetric states → many different phases

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  or symmetry protected topological (SPT) phase

- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm. Haldane 83
- 1 topo. ins. w/ $U(1) \times T$ symm. in 2D, Kane-Mele 05

Xiao-Gang Wen  Long-range entangled quantum matter and a unification of forces, matter, and information – a second quantum revolution
Symmetry protected topological (SPT) phases are gapped quantum phases with certain symmetry, which can be smoothly connected to the same trivial phase if we remove the symmetry.
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Group theory classifies 230 crystals. What classifies SPT orders?
SRE states with symmetry $\rightarrow$ SPT orders

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  ![Diagram](image)

- Group theory classifies 230 crystals. **What classifies SPT orders?**

- **A classification of (all?) SPT phase**: Chen-Gu-Liu-Wen 11
  
  For a system in $d$ spatial dimension with an on-site symmetry $G$, its SPT phases that do not break the symmetry $G$ are classified by the elements in $\mathcal{H}^{d+1}[G, U(1)]$ – the $d+1$ cohomology class of the symmetry group $G$ with $G$-module $U(1)$ as coefficient.
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- $H^{d+1}[G, U(1)]$ form an Abelian group: $a + b = c$,
  - Stacking $a$-SPT state and $b$-SPT state give us a $c$-SPT state.
### Bosonic SPT phases in any dim. and for any symmetry

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>$d = 0$</th>
<th>$d = 1$</th>
<th>$d = 2$</th>
<th>$d = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U(1) \times Z_2^T$ (top. ins.)</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2 (0)$</td>
<td>$\mathbb{Z}_2 (\mathbb{Z}_2)$</td>
<td>$\mathbb{Z}_2^2 (\mathbb{Z}_2)$</td>
</tr>
<tr>
<td>$U(1) \times Z_2^T \times \text{trans}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2^3$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2^8$</td>
</tr>
<tr>
<td>$U(1) \times Z_2^T$ (spin sys.)</td>
<td>$0$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$0$</td>
<td>$\mathbb{Z}_2^3$</td>
</tr>
<tr>
<td>$U(1) \times Z_2^T \times \text{trans}$</td>
<td>$0$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}_2^4$</td>
<td>$\mathbb{Z}_2^9$</td>
</tr>
<tr>
<td>$Z_2^T$ (top. SC)</td>
<td>$0$</td>
<td>$\mathbb{Z}_2 (\mathbb{Z}_2)$</td>
<td>$0 (0)$</td>
<td>$\mathbb{Z}_2 (0)$</td>
</tr>
<tr>
<td>$Z_2^T \times \text{trans}$</td>
<td>$0$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}_2^4$</td>
</tr>
<tr>
<td>$U(1)$</td>
<td>$\mathbb{Z}$</td>
<td>$0$</td>
<td>$\mathbb{Z}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$U(1) \times \text{trans}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}_2^4$</td>
</tr>
<tr>
<td>$Z_n$</td>
<td>$\mathbb{Z}_n$</td>
<td>$0$</td>
<td>$\mathbb{Z}_n$</td>
<td>$0$</td>
</tr>
<tr>
<td>$Z_n \times \text{trans}$</td>
<td>$\mathbb{Z}_n$</td>
<td>$\mathbb{Z}_n$</td>
<td>$\mathbb{Z}_n^2$</td>
<td>$\mathbb{Z}_n^4$</td>
</tr>
<tr>
<td>$D_{2h} = Z_2 \times Z_2 \times Z_2^T$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}_2^4$</td>
<td>$\mathbb{Z}_2^6$</td>
<td>$\mathbb{Z}_2^9$</td>
</tr>
<tr>
<td>$SO(3)$</td>
<td>$0$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td>$0$</td>
</tr>
<tr>
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<td>$0$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2^3$</td>
</tr>
</tbody>
</table>

Table of $\mathcal{H}^{d+1}[G, U_T(1)]$

- **“$Z_2^T$”**: time reversal, “trans”: translation, others: on-site symm.
- $0 \rightarrow$ only trivial phase.
- $(\mathbb{Z}_2) \rightarrow$ free fermion result

---

**Xiao-Gang Wen**

Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
Understand group cohomology through topological terms

- Consider an $d + 1$D system with symmetry $G$:
  
  $$S = \int d^d x dt \frac{1}{2\lambda} (\partial g(x^i, t))^2,$$
  
  symmetry $g(x) \rightarrow hg(x)$, $h, g \in G$

  If under RG, $\lambda \rightarrow \infty \rightarrow$ symmetric ground state described by a fixed point theory $S_{\text{fixed}} = 0$ or $e^{-S_{\text{fixed}}} = 1$.
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- Another $G$ symmetric system $S = \int d^d x dt \frac{1}{2\lambda} (\partial g(x^i, t))^2 + 2\pi i W$
  
  where $W[g(x^i, t)]$ is a topological term, which is classified by $\text{Hom}(\pi_{d+1}(G), \mathbb{Z})$. $\pi_{d+1}(G)$: mapping classes. $\text{Hom}()$: linear mapps.
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Understand group cohomology through topological terms

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- But in the $\lambda \rightarrow \infty$ limit, $g(x^i, t)$ is not a continuous function. The mapping classes $\pi_{d+1}(G)$ does not make sense. The above result is not valid. However, the idea is OK.
Understand group cohomology through topological terms

- Consider an \( d + 1 \)D system with symmetry \( G \):
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- Can we define topological terms and topological non-linear \( \sigma \)-models when space-time is a discrete lattice?
Lattice topological non-linear $\sigma$-model in 1+1D

- Generalize to space-time lattice $e^{-S} = \prod \nu(g_i, g_j, g_k)$, where $\nu(g_i, g_j, g_k) = e^{-\int \Delta L}$, with branched structure.
Lattice topological non-linear $\sigma$-model in 1+1D

- Generalize to space-time lattice: $e^{-S} = \prod \nu^{s(i,j,k)}(g_i, g_j, g_k)$, where $\nu^{s(i,j,k)}(g_i, g_j, g_k) = e^{-\int_\Delta L}$ and $s(i,j,k) = 1,*$

\[ \nu(\Delta L) = \prod_{i,j,k} \nu^{s(i,j,k)}(g_i, g_j, g_k) \]

\[ \nu = e^{-\int_\Delta L} \]

The solutions of the above equation are called group cocycle.

$\nu^2(g_0, g_1, g_2)$ and $\tilde{\nu}^2(g_0, g_1, g_2) = \nu^2(g_0, g_1, g_2) \beta_1(g_1, g_2) \beta_1(g_0, g_1) \beta_1(g_0, g_2)$ are both cocycles. We say $\nu^2 \sim \tilde{\nu}^2$ (equivalent).

The set of the equivalent classes of $\nu^2$ is denoted as $H_2[G, U(1)]$.  

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Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
Lattice topological non-linear $\sigma$-model in 1+1D

- Generalize to space-time lattice: $e^{-S} = \prod \nu^{s(i,j,k)}(g_i, g_j, g_k)$, where $\nu^{s(i,j,k)}(g_i, g_j, g_k) = e^{-\int_{\Delta} L}$ and $s(i, j, k) = 1,*$

- $\nu^{s(i,j,k)}(g_i, g_j, g_k)$ is a topological term if $\prod \nu^{s(i,j,k)}(g_i, g_j, g_k) = 1$ on any sphere, including a tetrahedron (simplest sphere).
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- On a tetrahedron $\rightarrow$ 2-cocycle condition

$$\nu(g_1, g_2, g_3)\nu(g_0, g_1, g_3)\nu^{-1}(g_0, g_2, g_3)\nu^{-1}(g_0, g_1, g_2) = 1$$

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Long-range entangled quantum matter and a unification of forces.
Group cohomology $\mathcal{H}^d[G, U(1)]$ in any dimensions

- **$d$-Cochain**: $U(1)$ valued function of $d + 1$ variables
  \[ \nu_d(g_0, ..., g_d) = \nu_d(gg_0, ..., gg_d) \in U(1), \quad \rightarrow \quad \text{on-site } G \text{-symmetry} \]

- **$\delta$-map**: $d + 1$ variable function $\nu_d \rightarrow d + 2$ variable function $(\delta \nu_d)$
  \[ (\delta \nu_d)(g_0, ..., g_{d+1}) = \prod_{i} \nu_d^{(-)}(g_0, ..., \hat{g}_i, ..., g_{d+1}) \]

- **Cocycles** = cochains that satisfy
  \[ (\delta \nu_d)(g_0, ..., g_{d+1}) = 1. \]

- **Equivalence relation** generated by any $d - 1$-cochain:
  \[ \nu_d(g_0, ..., g_d) \sim \nu_d(g_0, ..., g_d)(\delta \beta_{d-1})(g_0, ..., g_d) \]

- $\mathcal{H}^{d+1}[G, U(1)]$ is the equivalence class of cocycles $\nu_d$.

**Lattice topological non-linear $\sigma$-models with symmetry $G$ in $d$-spatial dimensions are classified by $\mathcal{H}^{d+1}[G, U(1)]$:**

\[ e^{-S} = \prod \nu_s^{(i,j,\ldots)}(g_i, g_j, \ldots), \quad \nu_{d+1}(g_0, g_1, ..., g_{d+1}) \in \mathcal{H}^{d+1}[G, U(1)] \]
As we change the lattice, the action amplitude $e^{-S}$ does not change:

$$\nu_2(g_0, g_1, g_2)\nu_2^{-1}(g_1, g_2, g_3) = \nu_2(g_0, g_1, g_3)\nu_2^{-1}(g_0, g_2, g_3)$$

$$\nu_2(g_0, g_1, g_2)\nu_2^{-1}(g_1, g_2, g_3)\nu_2(g_0, g_2, g_3) = \nu_2(g_0, g_1, g_3)$$

as implied by the cocycle condition:

$$\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)\nu_2^{-1}(g_0, g_2, g_3)\nu_2^{-1}(g_0, g_1, g_2) = 1$$

The topological non-linear $\sigma$-model is a RG fixed-point.
The ground state wave function $\Psi(\{g_i\}) = \prod_i \nu_2(g_i, g_{i+1}, g^*)$.

It is symmetric under the $G$-transformation $\Psi(\{g_i\}) = \Psi(\{gg_i\})$.

It is equivalent to a product state $|\Psi_0\rangle = \otimes_i \sum g_i |g_i\rangle$ under a LU transformation (note that $\Psi_0(\{g_i\}) = 1$).

The ground state is symmetric with a trivial topological order.
The ground state of the topological non-linear $\sigma$-model

\[ g_1 g_3 g_5 g_4 g^* = g_2 g_3 g_4 g^* = g_2 g_3 g_4 g^* \]

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The ground state of the topological non-linear $\sigma$-model

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- It is equivalent to a product state $|\Psi_0\rangle = \otimes_i \sum_{g_i} |g_i\rangle$ under a LU transformation (note that $\Psi_0(\{g_i\}) = 1$)

$$\Psi(\{g_i\}) = \prod_{i=\text{even}} \nu_2(g_i, g_{i+1}, g^*) \prod_{i=\text{odd}} \nu_2(g_i, g_{i+1}, g^*) \Psi_0(\{g_i\})$$

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Xiao-Gang Wen
Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
Summary: group cohomology $\rightarrow$ SPT states

How to probe the topological order and SPT order?
Summary: group cohomology → SPT states

How to probe the topological order and SPT order?

**Bulk topological phases ↔ Boundary anomalous theories**

SPT state with on-site symmetry
effective theory with anomalous symmetry
gauged SPT state
gauged theory with gauge anomaly
Boundary excitations of SPT phases

- SPT boundary excitations are described by a lattice non-linear $\sigma$-model at the boundary with a non-local Lagrangian term (a generalization of the WZW term for continuous $\sigma$-model):

$$e^{-\int_{\partial M^3} \mathcal{L}_{NLL}} = \prod_{\partial M^3} \nu_3^{s(i,j,k)}(g_i, g_j, g_k, g^*) \neq \prod_{\partial M^3} \mu_{\text{symm}}(g_i, g_j, g_k)$$

Either symmetric in one higher dimension or “non-symmetric” in the same dimension $\rightarrow$ discretized WZW term.
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either symmetric in one higher dimension or “non-symmetric” in the same dimension $\rightarrow$ discretized WZW term.

- **Conjecture** (proved for 1+1D boundary Chen-Liu-Wen 11): The boundary are gapless or degenerate: Chen-Liu-Wen 11; Xu 12; Senthil-Vishwanath 12; ...

  (a) if the boundary does not break the symmetry $\rightarrow$ gapless or topologically ordered (degenerate)

  (b) if the boundary break the symmetry $\rightarrow$ gapless or degenerate.

Generalize the result for WZW model in (1+1)D Witten 89.
Symmetry of the effective boundary theory

- Effective boundary symmetry in path-integral formalism:
  \[ e^{-\int_{\partial M_3} \mathcal{L}_{Bnd}} = \prod_{\partial M_3} \nu_3^{s(i,j,k)}(g_i, g_j, g_k, g^*) = \prod_{\partial M_3} \nu_3^{s(i,j,k)}(gg_i, gg_j, gg_k, g^*) \]
  but locally \[ \nu_3(gg_i, gg_j, gg_k, g^*) \neq \nu_3(g_i, g_j, g_k, g^*) \]

Under the symmetry transformation
\[ \mathcal{L}_{Bnd}[gg(x)] = \mathcal{L}_{Bnd}[g(x)] + df[g(x)]. \]
→ Anomalous symmetry
Symmetry of the effective boundary theory

- Effective boundary symmetry in path-integral formalism:

\[ e^{-\int_{\partial M^3} L_{\text{Bnd}}} = \prod_{\partial M^3} \nu_3^{s(i,j,k)}(g_i, g_j, g_k, g^*) = \prod_{\partial M^3} \nu_3^{s(i,j,k)}(gg_i, gg_j, gg_k, g^*) \]

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Under the symmetry transformation

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\[ \rightarrow \text{Anomalous symmetry} \]

- Effective boundary symmetry in Hamiltonian formalism \( \rightarrow \text{non-on-site symmetry} \)

\[ (e^{-\hat{H}_{\text{Bnd}}})\{gg'_i, \ldots\},\{gg_i, \ldots\} \neq (e^{-\hat{H}_{\text{Bnd}}})\{g'_i, \ldots\},\{g_i, \ldots\} \]

\[ = U_{\{g'_i, \ldots\}}^\dagger (e^{-\hat{H}_{\text{Bnd}}})\{g'_i, \ldots\},\{g_i, \ldots\} U_{\{g_i, \ldots\}} \]

where \( U_{\{g_i, \ldots\}} = \prod_{\langle ij \rangle} \nu_3(g_i, g_j, g^*, g^{-1}g^*) \)

\[ \hat{U}(g) = \prod \hat{U}_0(g) \prod \nu_3(g_i, g_j, g^*, g^{-1}g^*) \]
An example: $SU(2)$ SPT state in 2+1D Liu & Wen 12

For $SU(2)$, $\text{Hom}[\pi_3(SU(2)), \mathbb{Z}] = \mathcal{H}^3[SU(2), U(1)] = \mathbb{Z}$ → we can use the topological terms in field theory to classify the $SU(2)$-SPT states:

$$S_{\text{top}} = -i \frac{k}{12\pi} \int_M \text{Tr}(g^{-1}dg)^3, \quad k \in \mathbb{Z}, \quad g \in SU(2)$$

The $SU(2)$ symmetry $g(x) \to hg(x), \quad h, g(x) \in SU(2)$
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The $SU(2)$ symmetry $g(x) \rightarrow hg(x), \quad h, g(x) \in SU(2)$

• The edge excitations are gapless (described by fixed-point WZW):

$$S_{\text{Bnd}} = \int_{\partial M} \frac{k}{8\pi} \text{Tr}(\partial g^{-1}\partial g) - i \int_M \frac{k}{12\pi} \text{Tr}(g^{-1}dg)^3,$$

• At the fixed point, we have an equation of motion

$$\partial_z[(\partial_z g)g^{-1}] = 0, \quad \partial_z[(\partial_{\bar{z}} g^{-1})g] = 0, \quad z = x + it.$$

Right movers $[(\partial_z g)g^{-1}](z) \rightarrow SU(2)$-charges

Left movers $[(\partial_{\bar{z}} g^{-1})g](\bar{z}) \rightarrow SU_L(2)$-charges, $g(x) \rightarrow g(x)h_L$

Level-$k$ Kac-Moody algebra Witten 84
A classification of gauge anomalies

- The edge $SU(2)$ symmetry is anomalous (non-on-site): The above edge excitations cannot be described by a pure 1+1D lattice model with an on-site $SU(2)$ symmetry $U(g) = \bigotimes_i U_i(g)$.

- If we gauge the $SU(2)$ symmetry, we will get an anomalous chiral gauge theory on the edge, and an $SU(2)$ Chern-Simons theory of level-$k$ in the bulk.

- 2+1D $SU(2)$-SPT phases are classified by $\mathcal{H}^3(SU(2), U(1)) = \mathbb{Z}$

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In general and roughly speaking (after we gauge the symmetry),
$d + 1$D $G$-gauge-anomalies are classified $d + 2$D SPT phases, which is, in turn, classified by $\mathcal{H}^{d+1}(G, U(1))$

• $\mathcal{H}^{d+1}(G, U(1)) = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \rightarrow$ Adler-Bell-Jackiw anomalies.

• $\mathcal{H}^{d+1}(G, U(1)) = \mathbb{Z}_m \oplus \mathbb{Z}_n \oplus \cdots \rightarrow$ new global anomalies, for bosonic systems.
Two definitions of the gauge anomaly

**First definition:** non gauge invariance

- \[ S = \int d^d x \mathcal{L}(a_\mu, \psi) = \int d^d x \mathcal{L}(a_\mu + \partial_\mu f, e^{if} \psi), \text{ but} \]

\[ Z = \int D\psi DA_\mu \ e^{i \int d^d x \mathcal{L}(a_\mu, \psi)} \neq \int D\psi Da_\mu \ e^{i \int d^d x \mathcal{L}(a_\mu + \partial_\mu f, e^{if} \psi)} \]

- Example: 1+1D chiral SU(2) gauge theory:

\[ S = \int d^2 x \left[ \psi_R^\dagger (i\partial_t - a_0 - i\partial_x + a_x) \psi_R + \psi_L^\dagger (i\partial_t + i\partial_x) \psi_L + \frac{1}{\lambda} (F_{\mu\nu})^2 \right] \]
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- Anomalous gauge theory has no non-perturbative definition, say on lattice.
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**Second’ definition**

- Take \( \lambda \to 0 \) limit \( \to \) a theory with chiral \( SU(2) \) symmetry
  \( S = \int d^2 x \left[ \psi_R^*(i \partial_t - i \partial_x) \psi_R + \psi_L^*(i \partial_t + i \partial_x) \psi_L \right] \)
  \( \psi_R \to U \psi_R, \quad \psi_L \to \psi_L \).
- The above theory has no non-perturbative definition say on lattice, without breaking the symmetry.

Xiao-Gang Wen

Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
Lattice models and non-perturbative definition

\[ H = \sum_i \psi_i^\dagger \psi_{i+1} + h.c. \]

We can have a non-perturbative definition only if we break the \( SU(2) \) symmetry.

\[ SU(2) \]

trivial charged

\[ SU(2) \]

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**There is a way, without breaking the \( SU(2) \) symmetry!**

- Go to one higher dimension:
  - \( \nu = 1 \) QH state state for spin-up and spin-down fermions +
  - \( \nu = -1 \) QH state state for two spin-0 fermions,
  which is a non-trivial 2+1D symmetry protected topological (SPT) state protected by \( SU(2) \) symmetry.
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Summary

(fermionic) gauge anomaly ↔ (fermionic) anomalous symmetry (non-on-site symmetry) ↔ (fermionic) SPT state in one higher dimension ↔ group (super)-cohomology
A lattice definition of any anomaly-free gauge theories

- For anomaly-free gauge theories, the corresponding SPT state in one-higher dimensions is trivial $\sum \nu_i = 0 \rightarrow$

- the edge states can be gapped with no ground state degeneracy and symmetry breaking $\rightarrow$

- any anomaly-free gauge theories can be defined on lattices by considering interacting boson/fermion.
Try to define the $U(1) \times SU(2) \times SU(3)$ standard model

- After so many years of study, $U(1) \times SU(2) \times SU(3)$ standard model is not even a proper quantum theory, since we still do not have a non-perturbative definition of the model. (So it is not even a quantum model with a well defined $\hat{H}$.)
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Try to define a $U(1)$ chiral fermion model

- Let us try to put a 3+1D chiral fermion
  $$\hat{H} = \psi^\dagger (i \partial_i + A_i) \sigma^i \psi, \quad \psi = \text{two-component fermion operator}$$
on a 3D spatial lattice.

- We may set $A_i = 0$ and view $\hat{H}$ as a theory with a $U(1)$ symmetry.
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- **We cannot define the chiral fermion as a 3D free lattice model**
  - We can define the chiral fermion model as a boundary of
    a 4D gapped $U(1)$ symmetric free fermion lattice model

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Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
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  - We can define the chiral fermion model as a boundary of a 4D gapped $U(1)$ symmetric free fermion lattice model
  - The 4D free fermion lattice model is a non-trivial free fermionic $U(1)$ SPT state ($1 \in \mathbb{Z}$).
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  - We cannot gap out the mirror sector without breaking the \( U(1) \) symmetry. (Can be done by breaking the \( U(1) \) symmetry.)
Try to define a $U(1)$ chiral fermion model

- **We cannot define the chiral fermion as a 3D inter. lattice model**
  - We can define the chiral fermion model as a boundary of a 4D gapped $U(1)$ symmetric free/inter. fermion lattice model
  - The 4D fermion lattice model is a non-trivial inter. fermionic $U(1)$ SPT state (which induces AdAdA CS-term).
  - We cannot gap out the mirror sector without breaking the $U(1)$ symmetry, even with interactions. (Can be proved)
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  - The 4D fermion lattice model is a non-trivial inter. fermionic $U(1)$ SPT state (which induces $\text{AdAdA}$ CS-term).
  - We cannot gap out the mirror sector without breaking the $U(1)$ symmetry, **even with interactions**. (Can be proved)

- If we view the massless $U(1)$ chiral fermion as the boundary fermion of a 4+1D lattice, after we turn on the $U(1)$ gauge field, the massless gauge boson will live in the 4+1D bulk.
Try to define the $SO(10)$ chiral fermion model

- Try to put 16 chiral fermion in 3+1D
  \[ \hat{H} = \psi_\alpha^\dagger i \partial_i \sigma^i \psi_\alpha, \quad \psi_\alpha \text{ form the 16-dim. spinor rep. of } SO(10) \text{ on a 3D spatial lattice.} \]

- We cannot define the $SO(10)$ chiral fermion as a 3D free model
  - We can define the $SO(10)$ chiral fermion model as a boundary of a 4D gapped $SO(10)$ symmetric free fermion lattice model
  - The 4D free fermion lattice model is a non-trivial free fermionic $SO(10)$ SPT state.
  - We cannot gap out the mirror sector without breaking the $SO(10)$ symm. But can gap out the mirror sector by breaking the $SO(10)$ symm. \( \delta H = \psi_\alpha^T \epsilon n_a \Gamma^a_{\alpha\beta} \psi_\beta + h.c., \quad n_a \text{ form the 10-dim. rep. of } SO(10) \Gamma^a \text{ has } 8 \pm 1 \text{ eigenvalues and } 8 - 1 \text{ eigenvalues.} \)
Try to define the $SO(10)$ chiral fermion model

- We can define the $SO(10)$ chiral fermion as a 3D interacting fermion lattice model.
Try to define the $SO(10)$ chiral fermion model

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1. We gap out the mirror sector by breaking the $SO(10)$ symmetry:
   $$\delta H = \psi^T \epsilon n_a \Gamma^a_{\alpha\beta} \psi + h.c.$$ $n_a$ form the 10-dim. rep. of $SO(10)$

2. We let $n_a$ to have long-wave length fluctuations, to restore the $SO(10)$ symmetry, hopefully, the fluctuations do not kill the gap.

3. $n_a$ form a $S^9$ space with $\pi_d(S^9) = 0$ for $d = 0, \cdots, 4$. No zero-modes and, hopefully, not to kill the gap.
Try to define the $SO(10)$ chiral fermion model

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- We can define the $U(1) \times SU(2) \times SU(3)$ standard model as a 3D interacting fermionic lattice model with continuous time.
Topological states, anomalies, and the standard model

- **A classification of gapped quantum phases**
  
<table>
<thead>
<tr>
<th>$g_2$</th>
<th>$g_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SY−LRE 1</td>
<td>SY−LRE 2</td>
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<tr>
<td>SB−LRE 1</td>
<td>SB−LRE 2</td>
</tr>
<tr>
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</tr>
</tbody>
</table>
  
  - SET orders (tensor category w/ symmetry)
  - Symmetry breaking (group theory)
  - SPT orders (group cohomology theory)

- **Bulk topological phases ↔ Boundary anomalous theories**

  - SPT state with on-site symmetry
  - Effective theory with anomalous symmetry
  - Gauged SPT state
  - Theory with gauge anomaly

- **We can put the** $U(1) \times SU(2) \times SU(3)$ **standard model on lattice by simply allow fermions to interact**

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Xiao-Gang Wen

Long-range entangled quantum matter and a unification of force, matter, and information – a second quantum revolution
Can long-range entanglements → all wonders of universe?

**Seven wonders of universe:**
1. Identical particles
2. Gauge interactions
3. Fermi statistics
4. Chiral gauge coupling (Parity violation in weak interaction)
5. Small fermion mass $m_{\text{fermion}} \sim 10^{-18} M_{\text{Plank}}$
6. Lorentz symmetry
7. Quantum gravity

- **long-range entangled qubits** can naturally produce 1 - 5, maybe 6 and 7.

A unification of force and matter by quantum information via long-range entanglement
Long range entanglements
are source of many wonders

Xiao-Gang Wen

Long-range entangled quantum matter and a unification of forces